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E_{11} , Brane Dynamics and Duality Symmetries

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Abstract

Following hep-th/0412336 we use the non-linear realisation of the semi-direct product of E_{11} and its vector representation to construct brane dynamics. The brane moves through a spacetime which arises in the non-linear realisation from the vector representation and it contains the usual embedding coordinates as well as the world volume fields. The resulting equations of motion are first order in derivatives and can be thought of as duality relations. Each brane carries the full E_{11} symmetry and so the Cremmer-Julia duality symmetries. We apply this theory to find the dynamics of the IIA and IIB strings, the M2 and M5 branes, the IIB D3 brane as well as the one and two branes in seven dimensions.

1 Introduction

It was conjectured in 2001 that the underlying theory of strings and branes should have an E_{11} symmetry and this should be encoded in a non-linear realisation [1]. The precise conjecture being that the non-linear realisation of the semi-direct product of E_{11} with its vector representation, denoted $E_{11} \otimes_s l_1$, is the low energy effective action of strings and branes [2]. Although partial results on the construction of this non-linear realisation were found over the years it was only recently that it was shown that the equations of motion were essentially unique and were those of the maximal supergravity theories if one suitably restricted the fields and coordinates to be those at the lowest levels [3,4]. Furthermore it has been shown that the non-linear realisation $E_{11} \otimes_s l_1$ is a unified theory in that it contains all the maximal supergravity theories. The theories in the different dimensions arise from taking different decompositions of E_{11} and the gauged supergravity theories arise when one considers certain fields to have expectation values. For an review and the further references the reader is referred to reference [5]

The vector, or l_1 , representation that was used in the above non-linear realisation contains all the known brane charges [2,6,7,8] including those in low dimensions such as two and three. The low level elements of the vector representation begin with the charges of the simple branes which by definition have charges whose spacetime indices are totally antisymmetric, for example a simple p-brane has a charge that is of the form $Z^{a_1 \dots a_p}$. However, at higher levels one finds charges that carry a more complicated index structures, for example in eleven dimensions the vector representation has, in increasing order of level, the elements [2]

$$\begin{aligned}
& P_a, Z^{ab}, Z^{a_1 \dots a_5}, Z^{a_1 \dots a_7, b}, Z^{a_1 \dots a_8}, Z^{b_1 b_2 b_3, a_1 \dots a_8}, Z^{(cd), a_1 \dots a_9}, Z^{cd, a_1 \dots a_9}, \\
& Z^{c, a_1 \dots a_{10}} (2), Z^{a_1 \dots a_{11}}, Z^{c, d_1 \dots d_4, a_1 \dots a_9}, Z^{c_1 \dots c_6, a_1 \dots a_8}, Z^{c_1 \dots c_5, a_1 \dots a_9}, Z^{d_1, c_1 c_2 c_3, a_1 \dots a_{10}}, (2), \blacksquare \\
& Z^{c_1 \dots c_4, a_1 \dots a_{10}}, (2), Z^{(c_1 c_2, c_3), a_1 \dots a_{11}}, Z^{c, a_1 a_2}, (2), Z^{c_1 \dots c_3, a_1 \dots a_{11}}, (3), \dots
\end{aligned} \tag{1.1}$$

The blocks of indices contain indices that are totally antisymmetrised while $()$ indicates that the indices are symmetrised. The elements have multiplicity one except when there is a bracket after the object which contains a number that gives the multiplicity. All the generator belong to irreducible representations of $SL(11)$, for example $Z^{a_1 \dots a_7, b}$ obeys the constraint $Z^{[a_1 \dots a_7, b]} = 0$.

The form charges in the l_1 representation, that is, charges that have a single block of totally antisymmetrised indices can be readily computed and are listed in the table below [7,8,9].

Table 1. The form generators in the l_1 representation in D dimensions

D	G	Z	Z^a	$Z^{a_1 a_2}$	$Z^{a_1 \dots a_3}$	$Z^{a_1 \dots a_4}$	$Z^{a_1 \dots a_5}$	$Z^{a_1 \dots a_6}$	$Z^{a_1 \dots a_7}$
8	$SL(3) \otimes SL(2)$	$(\mathbf{3}, \mathbf{2})$	$(\bar{\mathbf{3}}, \mathbf{1})$	$(\mathbf{1}, \mathbf{2})$	$(\mathbf{3}, \mathbf{1})$	$(\bar{\mathbf{3}}, \mathbf{2})$	$(\mathbf{1}, \mathbf{3})$ $(\mathbf{8}, \mathbf{1})$ $(\mathbf{1}, \mathbf{1})$	$(\mathbf{3}, \mathbf{2})$ $(\mathbf{6}, \mathbf{2})$	$(\mathbf{6}, \mathbf{1})$ $(\mathbf{18}, \mathbf{1})$ $(\mathbf{3}, \mathbf{1})$ $(\mathbf{6}, \mathbf{1})$ $(\mathbf{3}, \mathbf{3})$
7	$SL(5)$	$\mathbf{10}$	$\bar{\mathbf{5}}$	$\mathbf{5}$	$\bar{\mathbf{10}}$	$\mathbf{24}$ $\mathbf{1}$	$\mathbf{40}$ $\mathbf{15}$ $\mathbf{10}$	$\mathbf{70}$ $\mathbf{50}$ $\mathbf{45}$ $\mathbf{5}$	- - - -
6	$SO(5, 5)$	$\bar{\mathbf{16}}$	$\mathbf{10}$	$\mathbf{16}$	$\mathbf{45}$ $\mathbf{1}$	$\bar{\mathbf{144}}$ $\mathbf{16}$	$\mathbf{320}$ $\mathbf{126}$ $\mathbf{120}$	- - -	- - -
5	E_6	$\bar{\mathbf{27}}$	$\mathbf{27}$	$\mathbf{78}$ $\mathbf{1}$	$\bar{\mathbf{351}}$ $\bar{\mathbf{27}}$	$\mathbf{1728}$ $\mathbf{351}$ $\mathbf{27}$	- - -	- - -	- - -
4	E_7	$\mathbf{56}$	$\mathbf{133}$ $\mathbf{1}$	$\mathbf{912}$ $\mathbf{56}$	$\mathbf{8645}$ $\mathbf{1539}$ $\mathbf{133}$ $\mathbf{1}$	- - - -	- - - -	- - - -	- - - -
3	E_8	$\mathbf{248}$ $\mathbf{1}$	$\mathbf{3875}$ $\mathbf{248}$ $\mathbf{1}$	$\mathbf{147250}$ $\mathbf{30380}$ $\mathbf{3875}$ $\mathbf{248}$ $\mathbf{1}$	- - - - -	- - - - -	- - - - -	- - - - -	- - - - -

At level zero the l_1 representation has the usual spacetime translations P_a . However, at level one we find coordinates which are scalars under the $SL(D)$ transformations of our usual spacetime, and so also Lorentz transformations, but belong to non-trivial representations of E_{11-D} . In particular, examining the table we find that they belong to the

$$\mathbf{10}, \quad \bar{\mathbf{16}}, \quad \bar{\mathbf{27}}, \quad \mathbf{56}, \quad \text{and} \quad \mathbf{248} \oplus \mathbf{1}, \quad \text{of} \quad SL(5), \quad SO(5, 5), \quad E_6, \quad E_7 \quad \text{and} \quad E_8 \quad (1.2)$$

for $D = 7, 6, 5, 4$ and 3 dimensions respectively [8,10].

Looking at equation (1.1) we see that, except for those at low levels, the branes charges in the vector representation of E_{11} generically have a more complicated index structure in that they have more than one block of totally antisymmetrised indices. We will refer to such branes as exotic, that is, are not simple branes. The existence of exotic branes was first observed in reference [10] which considered the U duality multiplets that contain some of the well known simple brane charges. In four and three dimensions, for example, they found that the brane charges belonged to rather large multiplets of the U duality

symmetry. These authors also postulated a formula for the tensions of the branes that they considered. Using this they were able surmise that if the branes could arise by some kind of dimensional reduction from some higher dimensional theory, which was unknown at that time, what would be the index structure of the corresponding brane charges in the higher dimensional theory. In doing so they came across exotic brane charges. The multiplets found in reference [10] were later shown to be contained in the vector representation of E_{11} [8,9].

Thus E_{11} and in particular its l_1 representation, predicts the existence of a very large, in fact an infinite number, of new branes charges and so branes whose physical role has yet to be properly identified [2,7,8,9]. A discussion of how to find what brane charges are contained in the vector representation, including explicit examples such as the point particle, string and membrane multiplets in various dimensions was given in reference [7]. Each brane charge in the vector representation corresponds to a weight in this representation and it was shown how one can construct a tension from the weight and so a tension for the corresponding brane. This formula agrees with the tensions of the branes where previously known. The dependence on the string coupling could be read off from one of the components in the corresponding weight in the vector representation [7].

In addition to knowing the brane tensions one also has some knowledge of the corresponding solutions. Indeed, given any positive root α in E_{11} one can construct a specific E_{11} group element [8]

$$g = e^{-\frac{1}{\alpha^2}(\ln N)\alpha \cdot H} e^{(1-N)E_\alpha} \quad (1.3)$$

where N is a function whose form is not specified. From the group element one can read off the values of the fields and so a putative solution. Using low level roots in eleven and ten dimensions one finds all the half BPS branes in these dimensions. In fact one can apply this formula to find generic solutions for any positive E_{11} root and in any dimension. Thus one finds a large number of new putative solutions whose charges will not generically be those of simple branes [8]. This construction was generalised to a formula for a group element that depends on two positive roots in eleven [11] and ten dimensions [12]. These solutions for low level roots reproduced all the quarter BPS branes in these dimensions. A further generalisation to include three and more numbers of E_{11} roots was given in refence [11].

When a rigid symmetry group G of a quantum field theory is spontaneously broken to a subgroup H , Goldstone theorem tells us the number of resulting massless particles. Furthermore, the low energy action that describes these particles is almost always the non-linear realisation of the group G with local subgroup H . This was the approach used to compute the dynamics of pions in the early days of particle physics. In this application spacetime was introduced by hand as a variable that the fields in the non-linear realisation depended on. Much of the past literature on E_{11} has been devoted to constructing the $E_{11} \otimes l_1$ non-linear realisation to construct a field theory, see for example in references [3] and [4], and it is in this theory that the maximal supergravity theories are contained. In contrast to the non-linear realisations used to describe pion dynamics, the $E_{11} \otimes l_1$ non-linear realisation automatically encodes a spacetime as its coordinates arise as the coefficients of the generators of the vector representation as they occur in the group element used to construct the non-linear realisation.

However, one can also use non-linear realisations to derive brane dynamics. An incomplete list of some of these papers is given in reference [13]. The bosonic p-brane in D dimensions can be thought of as the non-linear realisation of $SO(1, D-1) \otimes T^D$, where T^D are generators in the vector representation, and the local subgroup is $SO(1, D-p-2) \otimes SO(1, p)$ [14]. To construct the brane dynamics in the presence of background gravity fields one only has to consider the non-linear realisation of the group $GL(1, D-1) \otimes_s T^D$ [15].

Reference [9] outlined how to construct brane dynamics using the non-linear realisation of $E_{11} \otimes_s l_1$ and in particular gave a partial construction of the dynamics of the M2 and M5 branes. In this approach the coordinates that arise from the generators of the vector representations describe the embedding of the brane in the background spacetime, but they are taken to be functions of the parameters that parameterize the brane world volume. Since the vector representations contains all brane charges the brane coordinates arise from the brane charges in a natural way in this construction. The coupling of the brane to the supergravity fields is also automatic as these fields occur in the non-linear realisation as the coefficients of the Borel sub algebra generators of in the E_{11} group element.

The $E_{11} \otimes_s l_1$ non-linear realisation was used to construct the dynamics of the IIA string [16] by taking the decomposition of E_{11} that leads to the IIA theory in ten dimensions. The result was a $SO(10,10)$ invariant formulation of the string whose coordinates belonged to the vector representation of $SO(10,10)$. The final result agreed with the previously found result of references [17] and [18]. Reference [19] realised that to extend this work to consider other types of branes and encode higher duality symmetries required a new way of thinking and more recently reference [20] also raised the question of how one could incorporate the well known Cremmer-Julia symmetries into the brane dynamics.

In this paper we will extend the discussion of reference [16] and further develop the $E_{11} \otimes_s l_1$ non-linear realisation as it can be applied to branes in section two. In section three we discuss the dynamics of IIA string and in section four the dynamics of the the M2 brane and the M5. In section five we construct the IIB string and the D3 brane. In section six we consider the one brane and two brane dynamics in seven dimensions. Finally, in section seven discuss some general features of brane dynamics that emerge from the non-linear realisation. The calculations in the IIB theory and in seven dimensions use the Cartan involution invariant subalgebra of $E_{11} \otimes_s l_1$ in the decomposition appropriate to these theories; these results will be published elsewhere [33].

The main aim of this paper is to further developing the $E_{11} \otimes_s l_1$ non-linear realisation in the hope that we may use it to compute brane dynamics for all the branes in E theory. As we show it does lead to dynamics of some of the well known branes and gives the dynamics of the new branes in seven dimensions. This approach automatically encodes the Cremmer-Julia symmetries as they appear at level zero in E_{11} .

2. General Formalism

We are interested in the semi-direct product of E_{11} with its vector representation l_1 , which we denoted by $E_{11} \otimes_s l_1$. The commutators of this algebra can be written in the form

$$[R^{\bar{\alpha}}, R^{\bar{\beta}}] = f^{\bar{\alpha}\bar{\beta}}{}_{\bar{\gamma}} R^{\bar{\gamma}}, \quad [R^{\bar{\alpha}}, l_A] = -(D^{\bar{\alpha}})_A{}^B l_B \quad (2.1)$$

where $R^{\bar{\alpha}}$ are the generators of E_{11} and l_A are the generators belonging to the vector (l_1)

representation. We assume that the l_1 generators commute. The matrices $(D^{\bar{\alpha}})_A{}^B$ are the representation matrices of the E_{11} algebra in the l_1 representation. In previous papers we have used $\underline{\alpha}$ for the indices of the E_{11} generators but in this paper we will use $\bar{\alpha}$.

An important part will be played by the Cartan involution invariant subalgebra of E_{11} which we denote by $I_c(E_{11})$. The Cartan involution I_c takes positive root generators to negative root generators and its action can be taken to be

$$I_c(R^{\bar{\alpha}}) = -R^{-\bar{\alpha}} \quad (2.2)$$

for any E_{11} root $\bar{\alpha}$. The Cartan involution subalgebra is generated by $S^{\bar{\alpha}} \equiv R^{\bar{\alpha}} - R^{-\bar{\alpha}}$.

A non-linear realisation is specified by the choice of an algebra together with a choice of subalgebra, referred to as the local subalgebra. The non-linear realisation which leads to brane dynamics is a non-linear realisation of $E_{11} \otimes_s l_1$ with a local subalgebra \mathcal{H} that is a subgroup of $I_c(E_{11})$. The different choices of subalgebra \mathcal{H} lead to the different branes.

The non-linear realisation is constructed from a group element $g \in E_{11} \otimes_s l_1$ and we have to construct an action, or set of equations of motion, that is invariant under the transformations

$$g \rightarrow g_0 g, \quad g_0 \in E_{11} \otimes_s l_1, \quad \text{as well as} \quad g \rightarrow g h, \quad h \in \mathcal{H} \quad (2.3)$$

The group element $g_0 \in E_{11} \otimes_s l_1$ is a rigid transformation, that is, it is a constant, while the group element h belongs to the local subalgebra \mathcal{H} and it is a local transformation whose precise meaning will be discussed just below.

We can write the group element g of the non-linear realisation in the form

$$g = g_l g_h g_E \quad (2.4)$$

In this equation g_E is in the Borel subgroup of E_{11} , the group element g_l is formed from the generators of the l_1 representation while the group element g_h belongs to $I_c(E_{11})$. We can write the individual group elements in the form

$$g_l = e^{z^A l_A}, \quad g_E = e^{A_{\bar{\alpha}} R^{\bar{\alpha}}}, \quad g_h = e^{\varphi \cdot S} \quad (2.5)$$

In this equation the parameters z^A are the coordinates of the background space-time and they depend on the coordinates of the brane world volume ξ^α . The $A_{\bar{\alpha}}$ are the E_{11} background fields, which include those of the maximal supergravity theories, and they depend on the coordinates of the background spacetime z^A . The fields φ also depend on ξ^α and by a local transformation we mean one that depends on ξ^α . Clearly, we may use the local subalgebra \mathcal{H} to set some of the φ fields to zero. The brane world volume coordinates include, at lowest level, the usual brane coordinates ξ^α but they may also contain the higher level coordinates, for example in eleven dimensions the next possible coordinates would be $\xi_{\alpha_1 \alpha_2}$. The presence of the higher level coordinates corresponds to the possibility that the brane moves not only in the usual spacetime but also part of the background spacetime z^A . For branes with no world volume fields we will not need the higher level brane world volume coordinates, but they seem to be required when world volume fields are present.

It is apparent from this way of constructing brane dynamics that a given brane will be invariant under all the $E_{11} \otimes_s l_1$ symmetries, but which of these symmetries are linearly realised and which are spontaneously broken, and so non-linearly realised, depends on the choice of local subalgebra \mathcal{H} which in turn depends on the brane we are studying. We note that every brane will automatically be invariant under all the supergravity E_{11-d} duality symmetries in d dimensions as these occur in E_{11} at the lowest level.

As we have mentioned the dynamics is just that invariant under the symmetries of equation (2.3) and the best method to find these equations is to consider the Cartan forms

$$\mathcal{V} = g^{-1}dg = \mathcal{V}_E + \mathcal{V}_l + \mathcal{V}_h, \quad (2.6)$$

where

$$\mathcal{V}_E = g_E^{-1}dg_E, \text{ and } \mathcal{V}_l = g_E^{-1}g_h^{-1}(g_l^{-1}dg_l)g_hg_E, \quad \mathcal{V}_h = g_E^{-1}(g_h^{-1}dg_h)g_E \quad (2.7)$$

Clearly \mathcal{V}_E belongs to the E_{11} algebra and are just the Cartan forms of E_{11} ; we can write them as

$$\mathcal{V}_E \equiv (dz^\Pi G_{\Pi, \underline{\alpha}} R^\alpha) \quad (2.8)$$

where the $G_{\Pi, \underline{\alpha}}$ just depend on the E_{11} background fields $A_{\underline{\alpha}}$. We can write

$$\mathcal{V}_l \equiv d\xi^\alpha \nabla_{\underline{\alpha}}^B z^A l_A = g_E^{-1}g_h^{-1}(dz^A l_A)g_hg_E = g_E^{-1}(d\xi^\alpha \nabla_{\underline{\alpha}} z^A l_A)g_E \equiv d\xi^\alpha \nabla_{\underline{\alpha}} z^\Pi E_{\Pi}^A l_A \quad (2.9)$$

where E_{Π}^A is defined by $g_E^{-1}dz \cdot l g_E \equiv dz^\Pi E_{\Pi}^A l_A$. This last object just depends on the E_{11} background fields and it is the vielbein in background spacetime with coordinates z^A while $\nabla_{\underline{\alpha}} z^A$ depends only on the coordinates z^A and the fields φ .

It is instructive to recall how the above non-linear realisations differs from that used to derive the low energy effective field theory describing the behaviour of strings and branes. In this case the local subgroup is $I_c(E_{11})$ and so we may choose the group element g_h to be the identity element and so there are no φ fields. There is no brane and so no brane parameters ξ^α . However, we do have the coordinates z^A and the fields $A_{\underline{\alpha}}$ which depend on these coordinates. As a result, the Cartan forms associated with the vector representation just contain the vielbein and so are functions of the E_{11} fields. The dynamics is essentially encoded in equations which are functions of the E_{11} Cartan forms \mathcal{V}_E and the vielbein is used to convert world to tangent indices on these objects. We recall that the construction leads to a field theories which when suitably truncated to low levels are the maximal supergravity theories.

We note that in the non-linear realisation used to construct branes, the vielbein defined below equation (2.9) and the Cartan forms of equation (2.8) just contain the E_{11} fields and they are the same as one finds in the non-linear realisation discussed in the paragraph just above. As a result computing the non-linear realisation for the brane dynamics, set out above, will also lead to the low energy effective action for strings and branes whose truncation contains the maximal supergravity theories. The expressions for \mathcal{V}_E and the vielbein, at low levels, can be found, for example, in references [3] and [27]. The fields φ only occur in the Cartan forms $\nabla_{\underline{\alpha}}^B z^A$, or equivalently $\nabla_{\underline{\alpha}} z^A$, associated with the vector

representation and the dynamics of the branes will consist of invariant equations among these later objects. It is these later equations that we will focus on deriving in this paper.

In this paper we will compute the brane dynamics in the absence of background fields and so we will take g_E to be the identity matrix. In this case the non-linear realisation we are constructing is for the algebra $I_c(E_{11}) \otimes_s l_1$ with local subgroup \mathcal{H} and so the group element has the form $g = g_l g_h$ and the Cartan forms are given by

$$\mathcal{V} = g^{-1}dg = \mathcal{V}_l + \mathcal{V}_h, \quad (2.10)$$

where

$$\mathcal{V}_l = g_h^{-1}(g_l^{-1}dg_l)g_h = g_h^{-1}(dz^A l_A)g_h \equiv \nabla_\alpha z^A l_A, \quad \mathcal{V}_h = (g_h^{-1}dg_h) \quad (2.11)$$

Under the rigid transformation $g_0 \in I_c(E_{11})$

$$g_l \rightarrow g_0 g_l g_0^{-1}, \quad \text{or equivalently} \quad dz^A l_A \rightarrow g_0 dz^A l_A g_0^{-1}, \quad g_h \rightarrow g_0 g_h \quad (2.12)$$

while under the local transformation $h \in \mathcal{H}$ we have

$$g_l \rightarrow g_l, \quad g_h \rightarrow g_h h \quad (2.13)$$

The Cartan forms are inert under the rigid g_0 transformations. but under the local $h \in \mathcal{H}$ transformations they transform as

$$\mathcal{V} \rightarrow h^{-1}\mathcal{V}h + h^{-1}dh \quad (2.14)$$

and in particular that

$$\nabla z^A l_A \rightarrow h^{-1}(\nabla z^A l_A)h, \quad \mathcal{V}_h \rightarrow h^{-1}\mathcal{V}_h h + h^{-1}dh \quad (2.15)$$

where $d\xi^\alpha \nabla_\alpha$. Using this equation it is straightforward to explicitly compute these transformations from the $E_{11} \otimes_s l_1$ algebra. The dynamics is a set of equations that are invariant under these transformations and as the $\nabla_\alpha z^A$ transform covariantly we are looking for equations constructed from these quantities that transform into each other. We will also demand that the equations are invariant under arbitrary reparameterisations of the brane world volume, that is, diffeomorphism in the parameters ξ^α .

The brane dynamics in the presence of the background fields can be readily found from the resulting equations. In particular, by using equation (2.9), we can reinstate their presence by introducing the vielbein in the way that this equation dictates, that is, make the replacement

$$\nabla_\alpha z^A \rightarrow \nabla_\alpha^B z^A \quad (2.16)$$

The vielbein that occurs in the non-linear realisation has been computed in several dimensions in reference [27]. We will not in this paper consider the construction of the Wess-Zumino term in the brane dynamics.

3. The IIA string

The IIA theory arises from the non-linear realisation of $E_{11} \otimes_s l_1$ when we decompose the E_{11} algebra in terms of the $SO(10,10)$ algebra that results from deleting the node ten in the E_{11} Dynkin diagram. The $SO(10,10)$ group is the T duality group of string theory. We recall that, at level zero, the non-linear realisation of $E_{11} \otimes_s l_1$ leads [21] precisely to Siegel theory [22,23] which contains the massless fields of the NS-NS sector of string theory. The extension to include the massless fields of the R-R sector of the IIA string theory was found by computing the level one part of the $E_{11} \otimes_s l_1$ non-linear realisation [24].

The method of constructing dynamics in a field theory from a non-linear realisation, usually uses the Cartan forms as they are inert under the rigid symmetries of the non-linear realisation. Indeed the construction of the E_{11} low energy effective action of strings and branes has been found using this method. However, there is another method that works instead with the object $M \equiv gI_c(g^{-1})$ which is inert under the local transformations. Generally this later method is less useful for deriving field theory results and indeed difficult to implement in the E_{11} context. Nonetheless the dynamics of the IIA string has already been constructed from the non-linear realisation using the quantity M [16]. The results agreed with that of reference [17].

In this section we will use the more conventional method to construct the dynamics of the IIA string using the method of Cartan forms. We will find that this procedure is quite different to the M method and has a number of usual and unexpected features. As such it is useful to introduce this method in a simple setting so as to clearly illustrate the unusual features before using the method to construct the dynamics of more complicated branes dynamics given later in this paper.

At level zero the E_{11} generators are $K^a{}_b$, R^{ab} and \tilde{R}_{ab} , while in the l_1 representation, we have the generators P_a and Q^a [19]. The commutators of the semi-direct product algebra $E_{11} \otimes_s l_1$ at level zero are given by [19]

$$[K^a{}_b, K^c{}_d] = \delta_b^c K^a{}_d - \delta_d^a K^c{}_b, \quad [K^a{}_b, R^{cd}] = \delta_b^c R^{ad} - \delta_b^d R^{ac}, \quad [K^a{}_b, \tilde{R}_{cd}] = -\delta_c^a \tilde{R}_{bd} + \delta_d^a \tilde{R}_{bc},$$

$$[R^{ab}, \tilde{R}_{cd}] = \delta_{[c}^a K^b{}_{d]}, \quad [R^{ab}, R^{cd}] = 0 = [\tilde{R}_{ab}, \tilde{R}_{cd}] \quad (3.1)$$

$$[K^c{}_b, P_a] = -\delta_a^c P_b, \quad [R^{ab}, P_c] = -\frac{1}{2}(\delta_c^a Q^b - \delta_c^b Q^a), \quad [\tilde{R}_{ab}, P_c] = 0, \quad (3.2)$$

$$[K^a{}_b, Q^c] = \delta_b^c Q^a, \quad [\tilde{R}_{ab}, Q^c] = \frac{1}{2}(\delta_a^c P_b - \delta_b^c P_a), \quad [R^{ab}, Q^c] = 0, \quad (3.3)$$

where $\underline{a}, \underline{b}, \dots = 0, 1, \dots, D-1$.

The Cartan involution acts on the generators in the following way

$$I_c(K^a{}_b) = -K^b{}_a, \quad I_c(R^{ab}) = -\tilde{R}_{ab} \quad (3.4)$$

and as a result the Cartan Involution invariant subalgebra, $I_c(SO(10,10))$ is generated by

$$J_{\underline{a}}{}^{\underline{b}} \equiv K^{\underline{a}}{}_{\underline{b}} - K^{\underline{b}}{}_{\underline{a}}, \quad S_{\underline{ab}} \equiv 2(R^{\underline{ab}} - \tilde{R}_{\underline{ab}}) \quad (3.5)$$

The $I_c(SO(10, 10))$ algebra is given by

$$\begin{aligned}
[J_{\underline{ab}}, J_{\underline{cd}}] &= \eta_{\underline{bc}} J_{\underline{ad}} - \eta_{\underline{ac}} J_{\underline{bd}} - \eta_{\underline{bd}} J_{\underline{ac}} + \eta_{\underline{bc}} J_{\underline{ad}} \\
[S_{\underline{ab}}, S_{\underline{cd}}] &= \eta_{\underline{bc}} J_{\underline{ad}} - \eta_{\underline{ac}} J_{\underline{bd}} - \eta_{\underline{bd}} J_{\underline{ac}} + \eta_{\underline{bc}} J_{\underline{ad}} \\
[J_{\underline{ab}}, S_{\underline{cd}}] &= \eta_{\underline{bc}} S_{\underline{ad}} - \eta_{\underline{ac}} S_{\underline{bd}} - \eta_{\underline{bd}} S_{\underline{ac}} + \eta_{\underline{bc}} S_{\underline{ad}}
\end{aligned} \tag{3.6}$$

By adopting suitable generators one sees that it is none other than the algebra $SO(10) \otimes SO(10)$. Their commutators with the l_1 representation are given by

$$\begin{aligned}
[J_{ab}, P_c] &= -2\eta_{c[a} P_{b]}, \quad [S_{ab}, P_c] = -2\delta_c^{[a} Q^{b]} \\
[J_{ab}, Q^c] &= -2\delta_{[a}^c Q_{b]}, \quad [S_{ab}, Q^c] = -2\delta_{[a}^c P_{b]}
\end{aligned} \tag{3.7}$$

We will first consider the brane in the absence of background fields and so we will take the non-linear realisation of $SO(10) \otimes SO(10) \otimes_s l_1$. The IIA string lives in ten dimensions but our considerations can easily be generalised to D dimensions and by consider the non-realisation of $SO(D, D) \otimes_s l_1$ and trivial increase in the range of our indices.

We choose the local subgroup of our non-linear realisation to be the subgroup whose algebra is given by

$$\mathcal{H} = \{J_{ab}, S_{ab}, J_{a'b'}, S_{a'b'}, L_{ab'}, \dots\} \tag{3.8}$$

where $a, b, \dots = 0, 1$ and $a', b', \dots = 2, \dots, 9$ and $L_{ab'} \equiv J_{ab'} - \epsilon_a^c S_{cb'}$. We note that the local subalgebra \mathcal{H} is not $SO(1, 1) \otimes SO(8) \otimes SO(1, 1) \otimes SO(8)$ as one might naively expect. The M method, used in reference [16], is rather insensitive to the precise choice of local subalgebra used in the non-linear realisation and it would not be sensitive to the different choice of local subalgebra.

Using the commutation relations of equation (3.6) we find that the commutators of the local subalgebra \mathcal{H} to be given by certain of the commutators of equation (3.6) as well as

$$[L_{ab'}, L_{cd'}] = 0, \quad [S, L_{cd'}] = -L_{cd'} \tag{3.9}$$

$$[S_{a'b'}, L_{cd'}] = -\eta_{b'd'} \varepsilon_c^e L_{ea'} + \eta_{a'd'} \varepsilon_a^e L_{eb'} \quad [J_{a'b'}, L_{cd'}] = +\eta_{b'd'} \varepsilon_c^e L_{ea'} - \eta_{a'd'} \varepsilon_a^e L_{eb'} \tag{3.10}$$

where $S = \varepsilon_{ab} S$.

The commutators of the generators of \mathcal{H} with the vector representation are given by equation (3.7) as well as

$$\begin{aligned}
[L_{ab'}, P_c] &= -\eta_{ac} P_{b'} + \epsilon_{ac} Q_{b'}, \quad [L_{ab'}, Q_c] = -\eta_{ac} Q_{b'} + \epsilon_{ac} P_{b'}, \quad [S, P_a] = \epsilon_{ab} Q^c, \quad [S, P_{a'}] = 0 \\
[L_{ab'}, P_{c'}] &= \eta_{b'c'} (P_a - \epsilon_a^c Q_c), \quad [L_{ab'}, Q_{c'}] = \eta_{b'c'} (Q_a - \epsilon_a^c P_c). \quad [S, Q_a] = \epsilon_{ab} P^c, \quad [S, Q_{a'}] = 0
\end{aligned} \tag{3.11}$$

It will prove useful to divide the generators of the l_1 representation into two sets which are given by

$$\{N_c^- \equiv P_c - \varepsilon_{cd} Q_d\} \tag{3.12}$$

and

$$\{N_c^+ \equiv P_c + \varepsilon_{cd} Q^d, P_{a'}, Q_{a'}\} \tag{3.13}$$

The commutators of these newly defined generators are those of equation (3.7) as well as

$$\begin{aligned} [L_{ab'}, N_c^-] &= 0, \quad [L_{ab'}, N_c^+] = -2(\eta_{ac}P_{b'} - \varepsilon_{ac}Q_{b'}), \quad [S, N_c^\pm] = \mp 2N_c^\pm \\ [L_{ab'}, P_{c'}] &= \eta_{b'c'}N_a^-, \quad [L_{ab'}, Q_{c'}] = -\eta_{b'c'}\epsilon_a^d N_d^-, \quad [S, P_{b'}] = 0, \quad [S, Q_{b'}] = 0 \end{aligned} \quad (3.14)$$

We observe that the generators of the vector representation contain an irreducible representation when decomposed into representations of \mathcal{H} , namely N_c^- . We observe that the compliment of this representation does not transform into itself.

As explained in section two, the non-linear realisation is constructed from the group element $g = g_E g_h$ which we can now write in the form

$$g = e^{x^a(\xi)P_a + y_a(\xi)Q^a} e^{-\varphi^{ab'}J_{ab'}} \quad (3.15)$$

where, using the local symmetry of equation (2.13), we have chosen to g_h to be of the above form. We note that the only generators of $I_c(SO(10, 10))$ which are not in the local subgroup \mathcal{H} can be taken to be $J_{ab'}$. Examining the group element we find that the string moves in the background spacetime with coordinates x^a and y_a but we take the world volume coordinates to be just the ξ^α .

The Cartan form $\mathcal{V} \equiv g^{-1}dg$ can be written in the form

$$\mathcal{V} = d\xi^\alpha (\nabla_\alpha x^a P_a + \nabla_\alpha y^a Q_a + \nabla_\alpha x^{a'} P_{a'} + \nabla_\alpha y_{a'} Q^{a'}) \quad (3.16)$$

Writing this in terms of the generators of equations (3.12) and (3.13) we find it becomes

$$\mathcal{V} = d\xi^\alpha (\mathcal{E}_{\alpha-}^a N_a^- + \mathcal{E}_{\alpha+}^a N_a^+ + \nabla_\alpha x^{a'} P_{a'} + \nabla_\alpha y_{a'} Q^{a'}) \quad (3.17)$$

where

$$\mathcal{E}_{\alpha\pm}^a \equiv \frac{1}{2}(\nabla_\alpha x^a \mp \varepsilon^{ab}\nabla_\alpha y_b) \quad (3.18)$$

Using equation (2.15) we find that under the transforms $h = 1 + \Lambda^{ab'}L_{ab'} + \Lambda S$ the variations of the Cartan forms is given by

$$\begin{aligned} \delta(\nabla_\alpha x^a) &= -\Lambda^{ab'}\nabla_\alpha x_{b'} - \nabla_\alpha y_{b'}\epsilon^{ac}\Lambda_c^{b'} - \Lambda\nabla_\alpha y_b\epsilon^{ba}, \\ \delta(\nabla_\alpha y^a) &= -\Lambda^{ab'}\nabla_\alpha y_{b'} - \nabla_\alpha x_{b'}\epsilon^{ac}\Lambda_c^{b'} - \Lambda\nabla_\alpha x_b\epsilon^{ba}, \\ \delta(\nabla_\alpha x^{b'}) &= \Lambda^{ab'}(\nabla_\alpha x_a - \epsilon^{ac}\nabla_\alpha y_c), \quad \delta(\nabla_\alpha y^{b'}) = \Lambda^{ab'}(\nabla_\alpha y_a - \epsilon^{ac}\nabla_\alpha x_c) \end{aligned} \quad (3.19)$$

When expressed in terms of the the variables in equation (3.17) the same result is given by

$$\begin{aligned} \delta\mathcal{E}_{\alpha+}^a &= 0, \quad \delta\mathcal{E}_{\alpha-}^a = -\Lambda^{ab'}\nabla_\alpha x_{b'} - \Lambda^{db'}\epsilon_{ad}\nabla_\alpha y_{b'}, \\ \delta(\nabla_\alpha x^{b'}) &= 2\Lambda^{ab'}\mathcal{E}_{\alpha+}^a, \quad \delta(\nabla_\alpha y^{b'}) = -2\epsilon_{ac}\Lambda^{ab'}\mathcal{E}_{\alpha+}^c \end{aligned} \quad (3.20)$$

Examining these variations it is readily apparent that a set of equations that is invariant under the transformations of \mathcal{H} , and so all the transformations of the non-linear realisation, is given by

$$2\mathcal{E}_{\alpha+}^a = \nabla_{\alpha}x^a - \varepsilon^{ab}\nabla_{\alpha}y_b = 0 = \nabla_{\alpha}x^{a'} = \nabla_{\alpha}y^{a'} \quad (3.21)$$

At first sight the equations (3.21) do not look like the equations for the motion of the string. However, by multiplying by the matrix $s_{\alpha}^a \equiv \nabla_{\alpha}x^a$, and its inverse, we can write the first equation in (3.21) as

$$\sqrt{-\gamma}\gamma^{\alpha\beta}\nabla_{\beta}x^a = -\epsilon^{\alpha\beta}\nabla_{\beta}y^a \quad \text{or equivalently} \quad \sqrt{-\gamma}\gamma^{\alpha\beta}\nabla_{\beta}y^a = -\epsilon^{\alpha\beta}\nabla_{\beta}x^a \quad (3.22)$$

where

$$\gamma_{\alpha\beta} = (s\eta s^T)_{\alpha\beta} \equiv \nabla_{\alpha}x^a\eta_{ab}\nabla_{\beta}x^b \quad \text{and} \quad \gamma = \det\gamma_{\alpha\beta} \quad (3.23)$$

As a result equation (3.21) implies that

$$\sqrt{-\gamma}\gamma^{\alpha\beta}\nabla_{\beta}x^a = -\epsilon^{\alpha\beta}\nabla_{\beta}y^a \quad \text{or equivalently} \quad \sqrt{-\gamma}\gamma^{\alpha\beta}\nabla_{\beta}y^a = -\epsilon^{\alpha\beta}\nabla_{\beta}x^a \quad (3.24)$$

These are still not obviously the equations of motion of the string as we have the derivatives $\nabla_{\alpha}x^a$ rather than $\partial_{\alpha}x^a$. However, we note that we can write

$$\gamma_{\alpha\beta} = \nabla_{\alpha}x^a\eta_{ab}\nabla_{\beta}x^b = \nabla_{\alpha}x^a\eta_{ab}\nabla_{\beta}x^b + \nabla_{\alpha}x^{a'}\eta_{a'b'}\nabla_{\beta}x^{b'} = \nabla_{\alpha}x^a\eta_{ab}\nabla_{\beta}x^b = \partial_{\alpha}x^a\eta_{ab}\partial_{\beta}x^b \quad (3.25)$$

The last line follows from the fact that the line before it is $SO(10,10)$ invariant and so the fields $\varphi^{ab'}$, associated with the generators $J_{ab'}$ will be absent. Put another way we could carry out a $J_{ab'}$ transformation to remove these fields and the object would be unchanged.

The same argument can be used on the other covariant derivatives that appear in equation (3.24) as these equations are obviously invariant under $SO(10,10)$ transformations and so we can also remove all the $\varphi^{ab'}$ terms by such a transformation. As a result we find the equations

$$\sqrt{-\gamma}\gamma^{\alpha\beta}\partial_{\beta}x^a = -\epsilon^{\alpha\beta}\partial_{\beta}y^a \quad \text{or equivalently} \quad \sqrt{-\gamma}\gamma^{\alpha\beta}\partial_{\beta}y^a = -\epsilon^{\alpha\beta}\partial_{\beta}x^a \quad (3.26)$$

Taking the derivative of the first equation we find the well known equation for the motion of the string. The brane dynamics in the background including the level zero the fields, which are just the massless fields of the NS-NS sector of the IIA string, can be found by making the replacement of equation (2.16) using the vielbein of reference [19].

The equations of motion (3.20) are a mixture of traditional equations and inverse Higgs conditions [25]. The latter are conditions that allow one to algebraically solve for some of the fields of the non-linear realisation in terms of some of the other fields. We have only one $I_c(SO(D, D))$ field in the non-linear realisation, namely $\varphi_{ab'}$. As we will see below, the condition $\sqrt{-\gamma}\gamma^{\alpha\beta}\nabla_{\beta}x^{a'} + \epsilon^{\alpha\beta}\nabla_{\beta}y^{a'} = 0$ allows us to solve for the combination $\sqrt{-\gamma}\gamma^{\alpha\beta}\partial_{\beta}x^{a'} + \epsilon^{\alpha\beta}\partial_{\beta}y^{a'} = 0$ in terms of $\varphi_{ab'}$. Once this has been solved there are no $I_c(SO(D, D))$ fields in the equations which are just a function of x^a and y^b which just transform under the rigid transformations of equation (2.12).

An unexpected feature of the above calculation is the choice of local subalgebra of equation (3.8) rather than the naively expected subalgebra $SO(1, 1) \otimes SO(1, 1) \otimes SO(D - 2) \otimes SO(D - 2)$. In the latter case one would have two fields of φ type corresponding to the generators $J_{ab'}$ and $S_{ab'}$, rather than one corresponding to just $J_{ab'}$. As we mentioned already the calculation of reference [16] is rather immune to the choice of subalgebra as it works with the quantity M , mentioned above, that is invariant under local transformations.

In view of the unusual way the brane dynamics appears in the non-linear realisation when one uses the Cartan forms it is instructive to examine in detail what happens at the linearised level. Using equations (2.15) and (3.6) we find, up to terms that are at most linear in the fields φ , that the Cartan forms of equation (3.16) are given by

$$\nabla_\alpha x^{\underline{a}} = \partial_\alpha x^{\underline{a}} + 2\partial_\alpha x^c \phi^{(1)}_{\underline{c}}{}^{\underline{a}} + 2\partial_\alpha y^c \phi^{(2)}_{\underline{c}}{}^{\underline{a}} + \dots \quad (3.27)$$

$$\nabla_\alpha y^{\underline{a}} = \partial_\alpha y^{\underline{a}} + 2\partial_\alpha y^c \phi^{(1)}_{\underline{c}}{}^{\underline{a}} + 2\partial_\alpha x^c \phi^{(2)}_{\underline{c}}{}^{\underline{a}} + \dots \quad (3.28)$$

if we were to take the local group element g_h to be of that most general form, that is, of the form, namely $g_h = 1 + (\phi^{(1)}_{\underline{ab}} J^{\underline{ab}} + \phi^{(2)}_{\underline{ab}} S^{\underline{ab}})$. This would be the case if we had not used the local subalgebra to set some of the fields in g_h to zero, whereupon the actual group element is of the form $g_h = e^{-\varphi^{ab'} J_{ab'}}$ as given in equation (3.15). To make the content of equations (3.27) and (3.28) apparent we choose static gauge, that is, $\partial_\alpha x^c = \delta_\alpha^c$. Examining the first equation of motion of (3.21), that is $\nabla_\alpha x^a - \epsilon^{ab} \nabla_\alpha y_b = 0$, we find that at zeroth order we must take

$$\partial_\alpha x^c = \delta_\alpha^c \quad \text{and} \quad \partial_\alpha y_a = -\epsilon_{\alpha a} \quad (3.29)$$

Taking $\underline{a} = a'$ in equations (3.23) and (3.24) we find that then at the lowest non-trivial level they become

$$\nabla_\alpha x^{a'} = \partial_\alpha x^{a'} + 2\phi^{(1)}_{\alpha}{}^{a'} - 2\epsilon_{\alpha c} \phi^{(2)ca'} + \dots \quad (3.30)$$

and

$$\nabla_\alpha y^{a'} = \partial_\alpha y^{a'} - 2\epsilon_{\alpha c} \phi^{(1)ca'} + 2\phi^{(2)}_{\alpha}{}^{a'} + \dots \quad (3.31)$$

where we have taken the general group element g_h but we with only $\phi^{(1)a'}_c$ and $\phi^{(2)a'}_c$ to be non-zero. We may rewrite equations (3.30) and (3.31) as

$$\nabla^\alpha x^{a'} - \epsilon^{\alpha\delta} \nabla_\delta y^{a'} = \partial^\alpha x^{a'} - \epsilon^{\alpha\delta} \partial_\delta y^{a'} + 4(\phi^{(1)\alpha a'} - \epsilon^{\alpha\delta} \phi^{(2)}_{\delta}{}^{a'}) + \dots \quad (3.32)$$

and

$$\nabla_\alpha x^{a'} + \epsilon^{\alpha\delta} \nabla_\delta y^{a'} = 0 + \dots \quad (3.33)$$

We observe that only the combination $\phi^{(1)\alpha a'} - \epsilon^{\alpha\delta} \phi^{(2)}_{\delta}{}^{a'}$ occurs in the equations of motion and that if we set the right-hand side of equation (3.32) to zero this quantity will be solved in terms of $\partial_\alpha x^{a'} - \epsilon^{\alpha\delta} \partial_\delta y^{a'}$. This is an example of the so called inverse Higgs effect. The orthogonal combination, that is $\phi^{(1)\alpha a'} + \epsilon^{\alpha\delta} \phi^{(2)}_{\delta}{}^{a'}$, does not appear in the equations of motion (3.21) and so we cannot solve for this combination in terms of any of the fields $\partial_\alpha x^{\underline{a}}$. Our choice of local subalgebra \mathcal{H} of equation (3.8) and in particular the inclusion of the generators $L_{ab'}$ ensures that the orthogonal combination $\phi^{(1)\alpha a'} + \epsilon^{\alpha\delta} \phi^{(2)}_{\delta}{}^{a'}$ arises in

the local subgroup and not in the group element g_h and so is automatically absent from the theory. This observation can be used to justify the choice of local subgroup. As a result the combination in equation (3.32) is proportional to $\varphi^{\alpha a'}$.

Thus we see that setting $\nabla_\alpha x^{a'} = 0 = \nabla_\alpha y^{a'}$, as we did in equation (3.21), has the effect of solving for the φ fields that arise in the group element g_h and also enforcing the equation

$$\nabla_\alpha x^{a'} + \epsilon^{\alpha\delta} \nabla_\delta y^{a'} + \dots = 0 \quad (3.34)$$

which we recognise as the linearised version of the equation of motion of equation (3.26) in static gauge. We will find that the above pattern that emerges for the string case also occurs for all the other branes we study in this paper.

4. Branes in eleven dimensions

In this section we will consider the dynamics of the M2 and the M5 branes.

4.1 The M2 brane

In this section we will construct the dynamics of the M2 brane in eleven dimensions using the non-linear realisation $E_{11} \otimes_s l_1$ as explained in section two. The dynamics of the M2 brane was found in the classic paper of reference [26] and a formulation involving the dual fields x^a and $x^{a_1 a_2}$ was given in reference [25]. A partial account of the dynamics of the M2 brane from the non-linear realisation $E_{11} \otimes_s l_1$ was given in reference [9].

The eleven dimensional theory emerges when we take the decomposition of E_{11} into $SL(11)$ which appears when we delete node eleven of the E_{11} Dynkin diagram. We will first constructing the brane in the absence of background fields and so we consider the non-linear realisation of the algebra $I_c(E_{11}) \otimes_s l_1$ which in the decomposition to $SL(11)$ has the generators

$$I_c(E_{11}) = \{J_{\underline{a}_1 \underline{a}_2}, S_{\underline{a}_1 \underline{a}_2 \underline{a}_3}, S_{\underline{a}_1, \dots, \underline{a}_6}, S_{\underline{a}_1, \dots, \underline{a}_8, \underline{b}}, \dots\} \quad (4.1.1)$$

where $\underline{a}_1, \underline{a}_2 \dots = 0, 1, \dots, 10$ while the generators of the vector representations are

$$l_1 = \{P_{\underline{a}}, Z^{\underline{a}_1 \underline{a}_2}, Z^{\underline{a}_1 \dots \underline{a}_5}, Z^{\underline{a}_1 \dots \underline{a}_8}, Z^{\underline{a}_1 \dots \underline{a}_7, \underline{b}}, \dots\} \quad (4.1.2)$$

The algebra the generators of equation (4.1.1) and (4.1.2) obey can be found in reference [28].

We choose the local subalgebra \mathcal{H} to be given by

$$\begin{aligned} \mathcal{H} = \{ & J_{a_1 a_2}, J_{a'_1 a'_2}, \hat{S} \equiv \epsilon^{a_1 a_2 a_3} S_{a_1 a_2 a_3}, L_{ab'} \equiv 2J_{ab'} + \epsilon_{ae_1 e_2} S^{e_1 e_2}_{b'}, S_{ab'_1 b'_2}, \\ & \hat{S}_{a'_1 a'_2 a'_3} \equiv S_{a'_1 a'_2 a'_3} + \frac{1}{3} \epsilon^{e_1 e_2 e_3} S_{e_1 e_2 e_3} a'_1 a'_2 a'_3, S_{b'_1 \dots b'_6}, S_{a_1 b'_1 \dots b'_5}, S_{a_1 a_2 b'_1 \dots b'_4}, \dots \} \end{aligned} \quad (4.1.3)$$

where $a_1, a_2, \dots = 0, 1, 2$ and $a'_1 a'_2, \dots = 3, 4, \dots, 10$. A motivation for this choice of local subalgebra will be given when we analyse the linearised theory at the end of this section.

As discussed in section two the non-linear realisation is constructed from the group element $g = g_l g_h$. The group element g_l is given by

$$g_l = e^{x^a P_a} e^{x_{a_1 a_2} Z^{a_1 a_2}} e^{x_{a_1 \dots a_5} Z^{a_1 \dots a_5}} \dots \quad (4.1.4)$$

while the group element $g_h = e^{\varphi \cdot S}$ can be chosen to take the form

$$g_h = e^{-\varphi_a{}^{b'} J^a{}_{b'}} \dots \quad (4.1.5)$$

We note that the generators which are in $I_c(E_{11})$ but not in \mathcal{H} can be chosen to be $J^a{}_{b'}$ up to generators of level two.

The algebra obeyed by the generators of \mathcal{H} of equation (4.3), up to commutators that involve level two generators, is given

$$\begin{aligned} [J_{ab}, L_{cd}] &= -\eta_{ac} L_{bd} + \eta_{bc} L_{ad}, \quad [L_{ab'}, L_{cd}] = 0, \quad [L_{ab'}, \hat{S}] = -6L_{ab'}, \\ [L_{ab'}, S_{dc'_1 c'_2}] &= -2\eta_{ad} \hat{S}_{b' c'_1 c'_2} + 2\epsilon_{ade} \eta_{b[c'_1} L^e{}_{c'_2]}, \quad [L_{ab'}, \hat{S}_{c'_1 c'_2 c'_3}] = 6\eta_{b'[c'_1} S_{|a|c'_2 c'_3]}, \\ [S^{ab'_1 b'_2}, \hat{S}_{c'_1 c'_2 c'_3}] &= 2S^{ab'_1 b'_2}{}_{c'_1 c'_2 c'_3} - 3\delta_{[c'_1 c'_2}^{} L^a{}_{c'_3]} \\ [\hat{S}, \hat{S}_{a'_1 a'_2 a'_3}] &= 6\hat{S}_{a'_1 a'_2 a'_3}, \quad [\hat{S}, S_{ab'_1 b'_2}] = 0, \quad [S_{a'_1 a'_2 a'_3}, S_{b'_1 b'_2 b'_3}] = 2S_{a'_1 a'_2 a'_3 b'_1 b'_2 b'_3} \\ [S^{ac'_1 c'_2}, S_{bd'_1 d'_2}] &= 2S^{ac'_1 c'_2}{}_{bd'_1 d'_2} - 2\delta_{d'_1 d'_2}^{c'_1 c'_2} J^a{}_b - 8\delta_{b[d'_1}^{a[c'_1} J^{c'_2]}{}_{d'_2]} \end{aligned} \quad (4.1.6)$$

The commutators involving J_{ab} and $J_{a'b'}$ are as one naively expects and as a result we have not written them down.

Rather than work with the generators of the l_1 representation as given in equation (4.1.2) it will be advantageous to instead use the following objects

$$P_{a'}, N_a^\pm, Z_{a_1 a_2}, Z_{ab'}, N_{a'_1 a'_2}^\pm, Z_{a_1 a_2 b'_1 b'_2 b'_3}, Z_{ab'_1 \dots b'_4}, \hat{Z}_{a'_1 \dots a'_5}, Z_{a_1 \dots a_5}, \dots \quad (4.1.7)$$

where

$$\begin{aligned} N_a^\pm &\equiv P_a \pm \frac{1}{2} \epsilon_{ae_1 e_2} Z^{e_1 e_2}, \quad N_{a'_1 a'_2}^\pm = Z_{a'_1 a'_2} \mp \frac{1}{3!} \epsilon^{e_1 e_2 e_3} Z_{e_1 e_2 e_3 a'_1 a'_2}, \\ \hat{Z}_{a'_1 \dots a'_5} &= Z_{a'_1 \dots a'_5} + \frac{1}{3!} \epsilon^{e_1 e_2 e_3} \left(\frac{5}{3} Z_{e_1 e_2 e_3 [a'_1 \dots a'_4, a'_5]} - Z_{e_1 e_2 e_3 a'_1 \dots a'_5} \right), \end{aligned}$$

The generators at level four in the vector representation have eight Lorentz indices and we include such a contribution in the last equation as it was useful in the higher level calculations not presented in this paper.

The generators of the the vector representation given in equation (4.1.7) have the following commutators with the generators of \mathcal{H} :

$$\begin{aligned} [L_{ab'}, P_{c'}] &= 2\eta_{b'c'} N_a^+, \quad [L_{ab'}, N_c^+] = 0, \quad [L_{ab'}, N_c^-] = -4\eta_{ac} P_{b'} + 4\epsilon_{acd} Z^{db'}, \\ [L_{ab'}, N_{c'_1 c'_2}^-] &= 0, \quad [L_{ab'}, N_{c'_1 c'_2}^+] = -8\eta_{b'[c'_1} Z_{c'_2]a} + 2\epsilon_a{}^{e_1 e_2} Z_{e_1 e_2 b' c'_1 c'_2} \\ [L_{ab'}, Z_{cd}] &= 2\eta_{b'd'} \epsilon_{ace} N_e^+ - 2\eta_{ac} N_{b'd'}^-, \\ [\hat{S}, N_c^\pm] &= 6N_c^\pm, \quad [\hat{S}, N_{c'_1 c'_2}^\pm] = 6N_{c'_1 c'_2}^\pm, \quad [\hat{S}, Z_{ab'}] = 0, \quad [\hat{S}, Z_{c_1 c_2 d'_1 d'_2 d'_3}] = 0, \\ [S_{ab'_1 b'_2}, P_{c'}] &= 2\eta_{c'[b'_1} Z_{b'_2]a}, \quad [S_{ab'_1 b'_2}, N_c^\pm] = \eta_{ac} N_{b'_1 b'_2}^\mp, \end{aligned}$$

$$\begin{aligned}
[S_{ab'_1b'_2}, N_{c'_1c'_2}^\mp] &= -2\delta_{c'_1c'_2}^{b'_1b'_2} N_a^\pm + Z_{ab'_1b'_2c'_1c'_2}^\pm, \quad [S_{ab'_1b'_2}, Z_{cd'}] = Z_{acb'_1b'_2d'} - 4\eta_{d'}[b'_1P_{b'_2}], \\
[\hat{S}_{a'_1a'_2a'_3}, P_{c'}] &= 3\delta_{[a'_1}^{\prime} N_{a'_2a'_3]}^-, \quad [\hat{S}_{a'_1a'_2a'_3}, N_c^+] = 0, \quad [\hat{S}_{a'_1a'_2a'_3}, N_c^-] = -\varepsilon^{ce_1e_2} Z_{e_1e_2a'_1a'_2a'_3}, \\
[\hat{S}_{a'_1a'_2a'_3}, N_{c'_1c'_2}^-] &= \hat{Z}_{a'_1a'_2a'_3c'_1c'_2}, \quad [\hat{S}_{a'_1a'_2a'_3}, Z_{bc'}] = Z_{ba'_1a'_2a'_3c'}^+, \\
[\hat{S}_{a'_1a'_2a'_3}, N_{c'_1c'_2}^+] &= -12\delta_{c'_1c'_2}^{[a'_1a'_2} P^{a'_3]} + \hat{Z}_{a'_1a'_2a'_3c'_1c'_2} + \dots
\end{aligned} \tag{4.1.8}$$

where $+\dots$ means higher level generators. We observe that the generators

$$N_a^+, N_{a'_1a'_2}^-, Z_{a'_1\dots a'_5}, \dots$$

form an irreducible representation of the local subgroup \mathcal{H} up to the level we have calculated.

The dynamics is just a set of equations that are invariant under the transformations of equation (2.15). The Cartan form \mathcal{V}_l can be expressed as

$$\mathcal{V}_l = (\nabla x^a P_a + \nabla x_{a_1a_2} Z^{a_1a_2} + \nabla x_{a_1\dots a_5} Z^{a_1\dots a_5} + \dots) \tag{4.1.9}$$

where where $\nabla \equiv d\xi^\alpha \nabla_\alpha$ and we will write the Cartan forms as forms. When written in terms of the generators of equation (4.1.7) The Cartan form \mathcal{V}_l takes the form

$$\mathcal{V} = \mathcal{E}_a^+ N_+^a + \mathcal{E}_a^- N_-^a + \mathcal{E}^{a'} P_{a'} + \mathcal{E}_{a'_1a'_2}^- N_{a'_1a'_2}^- + \mathcal{E}_{a'_1a'_2}^+ N_{a'_1a'_2}^+ + \mathcal{E}_{ab'} Z^{ab'} + \dots \tag{4.1.10}$$

$$\begin{aligned}
\mathcal{E}^{a'} &\equiv \nabla x^{a'}, \quad \mathcal{E}_a^\pm \equiv \frac{1}{2}(\nabla x^a \mp \epsilon^{ae_1e_2} \nabla x_{e_1e_2}), \\
\mathcal{E}_{a'_1a'_2}^\pm &\equiv \frac{1}{2}(\nabla x_{a'_1a'_2} \pm \epsilon^{e_1e_2e_3} \nabla x_{e_1e_2e_3a'_1a'_2}), \quad \mathcal{E}_{ab'} \equiv \nabla x_{ab'},
\end{aligned} \tag{4.1.11}$$

Under the transformation $h = 1 - (\Lambda S + \Lambda^{ab'} L_{ab'} + \Lambda^{ab'_1b'_2} S_{ab'_1b'_2} + \Lambda^{b'_1b'_2b'_3} S_{b'_1b'_2b'_3})$ we find, using equation (2.15), that the Cartan forms transform as

$$\begin{aligned}
\delta \mathcal{E}_a^- &= -2\Lambda^{ab'_1b'_2} \mathcal{E}_{b'_1b'_2}^+ + 6\Lambda \mathcal{E}_a^- \\
\delta \mathcal{E}_a^+ &= \mathcal{E}^{b'} \Lambda^{ab'} + 2\mathcal{E}^c{}_{b'} \epsilon_{dca} \Lambda^{db'} - 2\Lambda_a{}^{b'_1b'_2} \mathcal{E}_{b'_1b'_2}^- + 6\Lambda \mathcal{E}_a^+ \\
\delta(\mathcal{E}^{a'}) &= -4\mathcal{E}^{b-} \Lambda_b{}^{a'} - 12\Lambda_{c'_1c'_2a'} \mathcal{E}_{c'_1c'_2}^+ \\
\delta(\mathcal{E}^{ab'}) &= 4\epsilon_{dc}{}^a \mathcal{E}^{c-} \Lambda^{db'} + 8\mathcal{E}^{+c'b'} \Lambda_{c'}^a - 2\Lambda^{ad'b'} \mathcal{E}_{d'} \\
\delta \mathcal{E}_{a'_1a'_2}^+ &= \Lambda^{ca'_1a'_2} \mathcal{E}_c^- + 6\Lambda \mathcal{E}_{a'_1a'_2}^+ \\
\delta \mathcal{E}_{a'_1a'_2}^- &= -2\mathcal{E}_{e[a'_2]} \Lambda^e{}_{[a'_1]} + \Lambda^c{}_{a'_1a'_2} \mathcal{E}_c^+ + 3\Lambda_{c'a'_1a'_2} \nabla_\alpha x^c + 6\Lambda \mathcal{E}_{a'_1a'_2}^-
\end{aligned} \tag{4.1.12}$$

Examining the variations of the Cartan form of equation (4.1.12) one sees that one can preserve all the symmetries of the non-linear realisation by setting

$$\delta\mathcal{E}_a^- = 0 = \mathcal{E}^{a'} = \mathcal{E}_{a'_1 a'_2}^+ = \mathcal{E}_{ab'} \quad (4.1.13)$$

which are equivalent to the equations

$$\nabla_\alpha x^a + \epsilon^{ae_1 e_2} \nabla_\alpha x_{e_1 e_2} = 0, \text{ or equivalently, } \nabla_\alpha x_{a_1 a_2} - \frac{1}{2} \epsilon_{a_1 a_2 c} \nabla_\alpha x^c = 0 \quad (4.1.14)$$

$$\nabla_\alpha x^{a'} = \nabla_\alpha x_{ab'} = 0 \quad (4.1.15)$$

$$\nabla_\alpha x_{a'_1 a'_2} + \epsilon^{e_1 e_2 e_3} \nabla_\alpha x_{e_1 e_2 e_3 a'_1 a'_2} = 0 \quad (4.1.16)$$

At first sight these are not the familiar equations of motion for the M2 brane. However, as discussed later in section seven in the context of the dynamics of general branes, we may first write equations (4.1.14) as

$$\sqrt{-\gamma} \gamma^{\alpha\beta} \nabla_\beta x^a = \epsilon^{\alpha\beta\gamma} \nabla_\beta x^{ab} \nabla_\gamma x_b, \quad (4.1.17)$$

or equivalently,

$$\sqrt{-\gamma} \gamma^{\alpha\beta} \nabla_\beta x^{a_1 a_2} = \frac{1}{2} \epsilon^{\alpha\gamma_1 \gamma_2} \nabla_{\gamma_1} x_{b_1} \nabla_{\gamma_2} x_{b_2} \quad (4.1.18)$$

where

$$\gamma_{\alpha\beta} \equiv (s\eta s^T) = \nabla_\alpha x^a \eta_{ab} \nabla_\beta x^b \quad \text{and} \quad \gamma = \det \gamma_{\alpha\beta} \quad (4.1.19)$$

As a result and using equations (4.1.15) and (4.1.16) we may write the dynamical equations as

$$\sqrt{-\gamma} \gamma^{\alpha\beta} \nabla_\beta x^a = \epsilon^{\alpha\beta\gamma} \nabla_\beta x^{ab} \nabla_\gamma x_b \quad (4.1.20)$$

Taking into account that the only φ fields are those associated with the Lorentz rotations $J_{ab'}$ and using arguments similar to those at the end of section three we conclude that we may write the equations of motions as

$$\sqrt{-\gamma} \gamma^{\alpha\beta} \partial_\beta x^a = \epsilon^{\alpha\beta\gamma} \partial_\beta x^{ab} \partial_\gamma x_b \quad (4.1.20)$$

Acting on this equation with a derivative we recover the well known equation for the motion of the M2 brane. It would be interesting to carry out the analysis of the brane dynamics at higher levels in the coordinates and in particular to systematically incorporate the level three coordinates. In particular it would be useful to know if there is non-trivial information encoded in the higher level coordinates that is not contained in the equations for the lower level coordinates discussed above.

The brane in the presence of the background fields, including those of eleven dimensional supergravity, can be found by making the replacement of equation (2.1.16) using the veilbein given in reference [27].

It is instructive to carry out the linearised analysis of the equations of motion (4.1.14) and (4.1.15) in order to understand their content in more detail. We begin by computing

the Cartan forms when taking the group element g_h to have the most general form and so given by

$$g_h = e^{\phi_{ab}J^{ab} + \phi_{a_1a_2a_3}S^{a_1a_2a_3} + \phi_{a_1\dots a_6}S^{a_1\dots a_6} + \dots} \quad (4.1.21)$$

We can use the local subgroup \mathcal{H} to restrict the group element as indeed we have done above in arriving at the group element g_h of equation (4.1.5). We can recover the later by restricting the fields ϕ . However, it will be useful to consider a general group element g_h so we can motivate the choice of the local subgroup \mathcal{H} . Using equation (2.15) we find that

$$\nabla_\alpha x^{\underline{c}} = \partial_\alpha x^{\underline{c}} + 6\phi^{\underline{e}_1\underline{e}_2\underline{e}}\partial_\alpha x_{\underline{e}_1\underline{e}_2} + 12.30\phi^{\underline{e}_1\dots\underline{e}_5\underline{e}}\partial_\alpha x_{\underline{e}_1\dots\underline{e}_5} + 2\partial_\alpha x^{\underline{e}}\phi_{\underline{e}}^{\underline{c}} + \dots \quad (4.1.22)$$

$$\nabla_\alpha x^{\underline{c}_1\underline{c}_2} = \partial_\alpha x^{\underline{c}_1\underline{c}_2} - 3\phi^{\underline{c}_1\underline{c}_2\underline{e}}\partial_\alpha x_{\underline{e}} + \frac{5!}{2}\phi_{\underline{e}_1\underline{e}_2\underline{e}_3}\partial_\alpha x^{\underline{e}_1\underline{e}_2\underline{e}_3\underline{c}_1\underline{c}_2} + 4\partial_\alpha x^{\underline{e}[\underline{c}_2]\underline{c}_1}\phi_{\underline{e}}^{\underline{c}_1]} + \dots \quad (4.1.23)$$

$$\nabla_\alpha x^{\underline{c}_1\dots\underline{c}_5} = \partial_\alpha x^{\underline{c}_1\dots\underline{c}_5} - \phi^{[\underline{c}_1\underline{c}_2\underline{c}_3}\partial_\alpha x^{\underline{c}_4\underline{c}_5]} - 3\phi^{\underline{c}_1\dots\underline{c}_5\underline{e}}\partial_\alpha x_{\underline{e}} + 10\partial_\alpha x^{\underline{e}[\underline{c}_2\dots\underline{c}_5}\phi_{\underline{e}}^{\underline{c}_1]} + \dots \quad (4.1.24)$$

if we only keep terms which are at most linear in the ϕ fields.

If we adopt static gauge $\partial_\alpha x^c = \delta_\alpha^c$, then equation of motion (4.14) implies that $\partial_\alpha x_{a_1a_2} = \frac{1}{2}\epsilon_{\alpha a_1a_2}$ at lowest order. Keeping terms that are at most linear in either the x or ϕ fields, we find that the Cartan forms are given by

$$\nabla_\alpha x^{c'} - \epsilon_\alpha^{\beta e}\nabla_\beta x_e^{c'} = \partial_\alpha x^{c'} - \epsilon_\alpha^{\beta e}\partial_\beta x_e^{c'} + \dots \quad (4.1.25)$$

while

$$\nabla_\alpha x^{c'} + \epsilon_\alpha^{\beta e}\nabla_\beta x_e^{c'} = \partial_\alpha x^{c'} + \epsilon_\alpha^{\beta e}\partial_\beta x_e^{c'} + 2(2\phi_\alpha^{c'} + 3\epsilon_{\alpha e_1 e_2}\phi^{e_1 e_2 c'}) + \dots \quad (4.1.26)$$

The equations of motion (4.1.15) set the expressions in equations (4.1.25) and (4.1.26) to zero. We observe that they only contain the combination $2\phi_\alpha^{c'} + 3\epsilon_{\alpha e_1 e_2}\phi^{e_1 e_2 c'}$ and not the orthogonal expression $3\phi_\alpha^{c'} - 2\epsilon_{\alpha e_1 e_2}\phi^{e_1 e_2 c'}$. Ensuring that the latter combination is not contained in the group element g_h motivates our choice of local subalgebra of equation (4.1.3). One finds that the former combination is proportional to $\varphi_\alpha^{c'}$. Setting to zero equation (4.1.26) enables us to solve for the expression involving x 's in terms of the ϕ fields and it is an example of the inverse Higgs effect. Setting to zero equation (4.1.25) is the correct linearised equation of motion for the M2 branes and it agrees with equation (4.20) at the linearised level.

4.2 The M5 brane

As the five brane has a six dimensional world volume we take the indices to have the range $a, b, \dots = 0, 1, \dots, 5$ and $a', b', \dots = 6, \dots, 10$. The local subgroup \mathcal{H} is a subgroup of $I_c(E_{11})$ and we choose it to contain the generators

$$\mathcal{H} = \{J_{ab}, J_{a'b'}, L_{ab'} \equiv J_{ab'} + \frac{2}{5!}\epsilon_{ae_1\dots e_5}S^{e_1\dots e_5}b', S_{-a_1a_2a_3}, S \equiv \frac{1}{6!}\epsilon^{a_1\dots a_6}S_{a_1\dots a_6}, \dots\} \quad (4.2.1)$$

where we adopt the notation

$$X_{\pm a_1a_2a_3} = \frac{1}{2}(X_{-a_1a_2a_3} \pm \frac{1}{3!}\epsilon_{a_1a_2a_3b_1b_2b_3}X^{b_1b_2b_3}) \quad (4.2.2)$$

for any object $X_{a_1 a_2 a_3}$. These generators obey the algebra

$$[L_{ac'}, L_{bd'}] = 0, [S_{-a_1 a_2 a_3}, S_{-b_1 b_2 b_3}] = 0, [S, S_{-a_1 a_2 a_3}] = \frac{1}{2} S_{-a_1 a_2 a_3}, [S, L_{ab'}] = \frac{1}{2} L_{ab'}, \dots \quad (4.2.3)$$

as well as the expected commutators with J_{ab} and $J_{a'b'}$.

In the last section we took a preferred basis of the generators of the vector representation, and so the Cartan forms, however, here we will take a more direct approach and compute the variations of the Cartan form as they appear in equation (4.1.9) under the local subgroup transformations using equation (2.15). One finds that

$$\begin{aligned} \delta(\nabla x^a) &= 6\Lambda_+^{e_1 e_2 a} \nabla x_{e_1 e_2} + \frac{1}{2} \epsilon^{ae_1 \dots e_5} \Lambda \nabla x_{e_1 \dots e_5} + 2\nabla x^{c'} \Lambda_{c'}^a, \\ \delta(\nabla x^{a'}) &= 2(\nabla x^c + \epsilon^{ce_1 \dots e_5} \nabla x_{e_1 \dots e_5}) \Lambda_c^{a'} \\ \delta(\nabla x_{a_1 a_2}) &= -3\Lambda_{+a_1 a_2 e} \nabla x^e + \frac{5!}{2} \Lambda_{+e_1 e_2 e_3} \nabla x^{e_1 e_2 e_3 a_1 a_2} + \dots \\ \delta(\nabla x^{a_1 \dots a_5}) &= -\Lambda_+^{[a_1 a_2 a_3} \nabla x^{a_4 a_5]} + \frac{1}{2 \cdot 5!} \epsilon^{a_1 \dots a_5 e} \Lambda \nabla x_e + \frac{2}{5!} \epsilon^{a_1 \dots a_5 b} \Lambda_{bc'} \nabla x^{c'} + \dots \\ \delta(\nabla x^{a_1 \dots a_4 b'}) &= -\frac{2}{5!} \Lambda^{+[a_1 a_2 a_3} \nabla x^{a_4] b'} \end{aligned} \quad (4.2.4)$$

where $+\dots$ refer to higher level terms which we are not considering here and $\nabla = d\xi^\alpha \nabla_\alpha$. One can also carry out a variation of the Cartan forms with a general group element h and then impose the conditions $\Lambda_a^{c'} = -\frac{3}{2} \epsilon_{ae_1 \dots e_5} \Lambda^{e_1 \dots e_5 c'}$ as well as the self duality conditions on $\Lambda_{+a_1 a_2 a_3}$ to ensure that the transformation is in the local subalgebra. In particular we find that under the above transformations of the local subalgebra that

$$\delta \mathcal{E}^a = 0, \delta \mathcal{E}^{a'} = 2\mathcal{E}^b \Lambda_b^{a'}, \delta \mathcal{E}_{a_1 a_2} = -3\Lambda_{+a_1 a_2 b} \mathcal{E}^b + \dots \quad (4.2.5)$$

where

$$\mathcal{E}^a \equiv \nabla x^a + \epsilon^{ab_1 \dots b_5} \nabla x_{b_1 \dots b_5}, \mathcal{E}^{a'} = \nabla x^{a'}, \mathcal{E}_{a_1 a_2}^L \equiv \nabla x_{a_1 a_2} \quad (4.2.6)$$

We first consider the **linearised** theory. From equation (4.2.6) we find that we can adopt the conditions

$$\mathcal{E}_\alpha^a = 0, \mathcal{E}_\alpha^{a'} = 0 \quad (4.2.7)$$

as well as

$$\mathcal{E}_{[\alpha e_1 e_2]}^L = 0 \quad (4.2.8)$$

and, up to the adoption of higher level constraints, they are invariant. We are working at the linearised level as the last equation antisymmetrises indices whose transformation character is different.

To understand the meaning of the equations (4.2.7) and (4.2.8) for the linearised theory it is instructive to calculate the Cartan forms at the linearised level. For this we take static gauge $\partial_\alpha x^a = \delta_\alpha^a$, $\partial_\alpha x^{a_1 \dots a_5} = \frac{1}{5!} \epsilon^{\alpha a_1 \dots a_5}$ and keep only terms that are linear

in the x 's and the ϕ 's. One finds that **if** one takes the general group element of equation (4.1.21) and uses equations (4.1.22) and (4.1.23) that

$$\nabla_\alpha x^{a'} - \epsilon_\alpha^{\beta b_1 \dots b_4} \nabla_\beta x_{b_1 \dots b_4}^{a'} = \partial_\alpha x^{a'} - \epsilon^{\alpha \beta b_1 \dots b_4} \partial_\beta x_{b_1 \dots b_4}^{a'}, \quad (4.2.9)$$

$$\nabla_\alpha x^{a'} + \epsilon_\alpha^{\beta b_1 \dots b_4} \nabla_\beta x_{b_1 \dots b_4}^{a'} = \partial_\alpha x^{a'} + \epsilon^{\alpha \beta b_1 \dots b_4} \partial_\beta x_{b_1 \dots b_4}^{a'} + 4\hat{\phi}_\alpha^{a'}, \quad (4.2.10)$$

$$\nabla_\alpha x_{a_1 a_2} = \partial_\alpha x_{a_1 a_2} - 6\hat{\phi}_{-\alpha a_1 a_2} \quad (4.2.11)$$

where

$$\hat{\phi}_\alpha^{a'} = \frac{1}{2}(\phi_\alpha^{a'} + \frac{3}{2}\epsilon_{\alpha e_1 \dots e_5} \phi^{e_1 \dots e_5 a'}) \quad (4.2.12)$$

We have chosen our local subalgebra just so that the ϕ 's that occur in the Cartan forms contain the combination that occurs in equation (4.2.12) and not the orthogonal combination. As a result we take the generator $L_{ab'}$ to be in the local subalgebra \mathcal{H} . The analogous statement holds for the self-dual character of the generator $S_{-a_1 a_2 a_3}$ which belong to the local subalgebra \mathcal{H} . We observe that half of the equations $\mathcal{E}_\alpha^{a'} = 0$ and $\mathcal{E}_{[\alpha e_1 e_2]} = 0$ are equations of motion and half are inverse Higgs conditions.

We now present a proposal for the **non-linear** theory. We adopt equations (4.2.7) in the non-linear theory. Following the arguments in section seven we conclude that equation (4.2.7) can be written as

$$\sqrt{-\gamma} \gamma^{\alpha\beta} \nabla_\beta x^a = -\epsilon^{\alpha\beta_1 \dots \beta_5} \nabla_{\beta_1} x^{a b_1 \dots b_5} \nabla_{\beta_2} x_{b_2} \dots \nabla_{\beta_5} x_{b_5} \quad (4.2.13)$$

It is tempting to write equation (4.2.8) in the non-linear theory as $\nabla_{[a_1} x_{a_2 a_3]} = 0$ where $\nabla_e = (s^{-1})_e^\alpha \nabla_\alpha$. However this equation is not invariant under the local subalgebra transformations due to the transformation of $\delta((s^{-1})_e^\alpha \nabla_\alpha)$. To address this matter we will consider that the brane, which moves in the spacetime with coordinates x^a , $x_{a_1 a_2}, \dots$, has an enlarged world volume which is parameterised by $\xi^\alpha = \{\xi^\alpha, \xi^{\alpha_1 \alpha_2}\}$. This is natural given that it is moving in a much large spacetime that the usual eleven dimensional spacetime. We now consider the object

$$E_{\underline{\alpha}}^A = \begin{pmatrix} \nabla_\alpha x^a & \nabla_\alpha x^{a_1 a_2} \\ \nabla_{\alpha_1 \alpha_2} x^a & \nabla_{\alpha_1 \alpha_2} x^{a_1 a_2} \end{pmatrix} \quad (4.2.14)$$

In the non-linear theory we than replace equation (4.2.8) by

$$\mathcal{E}_{a_1 a_2 a_3} \equiv \nabla_{[a_1} x_{a_2 a_3]} + \frac{2}{3} \nabla^{cd} x_{c[a_1} \nabla_{d|a_1} x_{a_2 a_3]} = 0 \quad (4.2.15)$$

We note, using equation (4.2.4), that under the $L_{ab'}$ and $S_{-a_1 a_2 a_3}$ transformations of the local subalgebra

$$\delta(\nabla x^a) = -6\Lambda_+^{e_1 e_2 a} \nabla x_{e_1 e_2} = -6\Lambda_+^{e_1 e_2 a} E_{e_1 e_2} \quad (4.2.16)$$

where we have used the conditions of equation (4.2.7). As a result we find that

$$\delta(\nabla_{a_1 a_2}) = -6\Lambda_{+a_1 a_2}^c \nabla_c \quad (4.2.17)$$

where $\nabla_{a_1 a_2} \equiv (E^{-1})_{a_1 a_2}{}^\alpha \nabla_\alpha$. Hence we find that

$$\delta(\mathcal{E}_{a_1 a_2 a_3}) = -2\mathcal{E}_{[a_1|e_1 e_2} \Lambda_{+}{}^{e_1 e_2 d} \nabla_d x_{|a_1 a_2]} + \dots = 0 \quad (4.2.18)$$

which is invariant. We have used the condition of equation (4.2.15) and we have discarded the terms in the variation that contain derivatives with respect to the higher level world volume coordinates that parameterise the brane. The $+\dots$ indicates the presence of such terms.

The procedure we have used is similar to that used when constructing the field theory, that is, supergravity extended theories, in E theory. The problem can be traced to the fact that the variation of $\nabla_{[a_1} x_{a_2 a_3]}$ is not gauge invariant, however, this is addressed in equation (4.2.15) by the addition of the second term involving a derivative with respect to the higher level coordinate $\xi_{a_1 a_2}$. Thus we find the same connection, as in the field theory case, between gauge symmetry and the presence of the higher level coordinates. In the field theory one finds that such terms added are vital for the consistency of the theory even though the physical meaning of the higher level coordinates is still unclear. We present the above non-linear theory only as a proposal as it would be good to examine its consistency to the same extent as has been done in the field theory case before being sure that it is correct.

The part of equation (4.2.15) that is an equation of motion, rather than an inverse Higgs condition, and does not involve derivatives with respect to the higher level brane coordinates can be written in the non-linear theory as

$$\nabla_{[a_1} x_{a_2 a_3]} + \frac{1}{3!} \epsilon_{a_1 a_2 a_3}{}^{b_1 b_2 b_3} \nabla_{[b_1} x_{b_2 b_3]} = 0 \quad (4.2.19)$$

While this appears to be quite a simple equation its content is only apparent once one solves the inverse Higgs condition which obeys a similar condition but with the opposite duality. It would be interesting to see if the resulting equation which involves x^a and the gauge field x_a^j agrees with the known dynamics for the M5 brane [35].

5. Branes in the IIB theory

In this section we will derive the equations of motion of the F1 and D1, strings as well as those for the D3 brane that occur in the IIB theory. The IIB theory results from deleting node nine in the E_{11} Dynkin diagram and decomposing the $E_{11} \otimes_s l_1$ algebra in terms of the subalgebra of the remaining nodes, that is, $SL(10) \otimes SL(2)$ [34]. The IIB algebra in this decomposition can be found in reference [27]. We will compute the dynamics in the absence of the background IIB supergravity fields and so we will consider the non-linear realisation of the semi-direct product of the Cartan involution subalgebra $I_c(E_{11})$ with the vector representation, that is, $I_c(E_{11}) \otimes_s l_1$. This algebra was worked out in collaboration with Michaella Pettit [33]. At level zero $I_c(E_{11})$ is $SO(1,9) \otimes SO(2)$. The generators of $I_c(E_{11})$ in this decomposition are

$$J_{\underline{a}\underline{b}}, S, S_i^{\underline{a}_1 \underline{a}_2}, S^{\underline{a}_1 \dots \underline{a}_4}, S_i^{\underline{a}_1 \dots \underline{a}_6}, S_{i_1 i_2}^{\underline{a}_1 \dots \underline{a}_8}, S^{\underline{a}_1 \dots \underline{a}_7, \underline{b}}, \dots \quad (5.1)$$

where $i = 1, 2$ and $\underline{a} = 0, 1, \dots, 9$. Their definitions in terms of the underlying E_{11} generators are given in equation (A.2) and $S_{i_1 i_2}^{\underline{a}_1 \dots \underline{a}_8} = S_{(i_1 i_2)}^{\underline{a}_1 \dots \underline{a}_8}$. The generators of the l_1 representations in this decomposition are given by

$$P_{\underline{a}}; \quad Z_{\underline{i}}^{\underline{a}}; \quad Z^{\underline{a}_1 \underline{a}_2 \underline{a}_3}; \quad Z_{\underline{i}}^{\underline{a}_1 \dots \underline{a}_5}; \quad Z_{\underline{i} \underline{j}}^{\underline{a}_1 \dots \underline{a}_7}; \quad Z^{\underline{a}_1 \dots \underline{a}_7}, \quad Z^{\underline{a}_1 \dots \underline{a}_6, \underline{b}}, \dots \quad (5.2)$$

We raise and lower the i, j, \dots indices with $\delta_{i,j}$ but the indices $\underline{a}, \underline{b}, \dots$ with the Minkowski metric corresponding to the fact that we are working with the algebra $SO(1, 9) \otimes SO(2)$ at lowest level.

The group element used to construct the non-linear realisation is of the form $g = g_l g_h$ where

$$g_l = e^{x^{\underline{a}} P_{\underline{a}} + x_{\underline{a}}^i Z_{\underline{i}}^{\underline{a}} + x_{\underline{a}_1 \underline{a}_2 \underline{a}_3} Z^{\underline{a}_1 \underline{a}_2 \underline{a}_3} + x_{\underline{a}_1 \dots \underline{a}_5}^i Z_{\underline{i}}^{\underline{a}_1 \dots \underline{a}_5} + x_{\underline{a}_1 \dots \underline{a}_7}^{ij} Z_{\underline{i} \underline{j}}^{\underline{a}_1 \dots \underline{a}_7} + x_{\underline{a}_1 \dots \underline{a}_7} Z^{\underline{a}_1 \dots \underline{a}_7} + x_{\underline{a}_1 \dots \underline{a}_6, \underline{b}} Z^{\underline{a}_1 \dots \underline{a}_6, \underline{b}} + \dots} \quad (5.3)$$

while g_h belongs to the local subalgebra which depends on the brane being studied.

The Cartan forms can be written as

$$\begin{aligned} \mathcal{V}_l = g_h^{-1} (dx^A l_A) g_h = & \nabla x^{\underline{a}} P_{\underline{a}} + \nabla x_{\underline{a}}^i Z_{\underline{i}}^{\underline{a}} + \nabla x_{\underline{a}_1 \underline{a}_2 \underline{a}_3} Z^{\underline{a}_1 \underline{a}_2 \underline{a}_3} \\ & + \nabla x_{\underline{a}_1 \dots \underline{a}_5}^i Z_{\underline{i}}^{\underline{a}_1 \dots \underline{a}_5} + \nabla x_{\underline{a}_1 \dots \underline{a}_7}^{ij} Z_{\underline{i} \underline{j}}^{\underline{a}_1 \dots \underline{a}_7} + \nabla x_{\underline{a}_1 \dots \underline{a}_7} Z^{\underline{a}_1 \dots \underline{a}_7} + \nabla x_{\underline{a}_1 \dots \underline{a}_6, \underline{b}} Z^{\underline{a}_1 \dots \underline{a}_6, \underline{b}} + \dots \end{aligned} \quad (5.4)$$

where $\nabla = d\xi^\alpha \nabla_\alpha$

Using equation (2.15) the transformations of the Cartan forms under a local transformation which involves the most general possible $h \in I_c(E_{11})$, that is, $h = 1 - (-J_{\underline{a}_1 \underline{a}_2} \Lambda^{\underline{a}_1 \underline{a}_2} + S_{\underline{i}}^{\underline{a}_1 \underline{a}_2} \Lambda_{\underline{a}_1 \underline{a}_2}^{\underline{i}} + S^{\underline{a}_1 \dots \underline{a}_4} \Lambda_{\underline{a}_1 \dots \underline{a}_4} + \Lambda S, \mathcal{V}_l)$ are given found to be

$$\delta \mathcal{V}_l = [-J_{\underline{a}_1 \underline{a}_2} \Lambda^{\underline{a}_1 \underline{a}_2} + S_{\underline{i}}^{\underline{a}_1 \underline{a}_2} \Lambda_{\underline{a}_1 \underline{a}_2}^{\underline{i}} + S^{\underline{a}_1 \dots \underline{a}_4} \Lambda_{\underline{a}_1 \dots \underline{a}_4} + \Lambda S, \mathcal{V}_l] \quad (5.5)$$

which results, using reference [33] in the variations

$$\begin{aligned} \delta(\nabla_\alpha x^{\underline{a}}) &= -4\Lambda_{\underline{i}}^{\underline{a} \underline{b}} \nabla_\alpha x_{\underline{b}}^{\underline{i}} + 2\nabla_\alpha x^{\underline{b}} \Lambda_{\underline{b}}^{\underline{a}} + 48\nabla_\alpha x^{\underline{b}_1 \underline{b}_2 \underline{b}_3} \Lambda_{\underline{b}_1 \underline{b}_2 \underline{b}_3}^{\underline{a}} \\ \delta(\nabla_\alpha x_{\underline{a}}^{\underline{i}}) &= -\Lambda_{\underline{i}}^{\underline{a} \underline{b}} \nabla_\alpha x_{\underline{b}}^{\underline{i}} + 2\nabla_\alpha x_{\underline{b}}^{\underline{i}} \Lambda_{\underline{b}}^{\underline{a}} - 6\nabla_\alpha x^{\underline{b}_1 \underline{b}_2 \underline{a}} \epsilon_{ij} \Lambda_{\underline{b}_1 \underline{b}_2}^{\underline{j}} + 120\nabla_\alpha x_{\underline{b}_1 \dots \underline{b}_4 \underline{a}}^{\underline{i}} \Lambda_{\underline{b}_1 \dots \underline{b}_4}^{\underline{a}} - \frac{1}{2} \nabla_\alpha x_{\underline{a}}^{\underline{j}} \epsilon_{\underline{j}}^{\underline{i}} \Lambda \\ \delta(\nabla_\alpha x_{\underline{a}_1 \underline{a}_2 \underline{a}_3}) &= 6\nabla_\alpha x_{\underline{b} [\underline{a}_2 \underline{a}_3]} \Lambda_{\underline{a}_1}^{\underline{b}} + 2\nabla_\alpha x^{\underline{b}} \Lambda_{\underline{b} \underline{a}_1 \underline{a}_2 \underline{a}_3} - 20\nabla_\alpha x_{\underline{b}_1 \underline{b}_2 \underline{a}_1 \underline{a}_2 \underline{a}_3}^{\underline{i}} \Lambda_{\underline{i}}^{\underline{b}_1 \underline{b}_2} - \nabla_\alpha x_{[\underline{a}_1}^{\underline{j}} \epsilon_{\underline{j} \underline{a}_2 \underline{a}_3]}^{\underline{i}} \Lambda_{\underline{a}_1}^{\underline{i}} \\ \delta(\nabla_\alpha x_{\underline{a}_1 \dots \underline{a}_5}^{\underline{i}}) &= 10\nabla_\alpha x_{\underline{b} [\underline{a}_2 \dots \underline{a}_5]}^{\underline{i}} \Lambda_{\underline{a}_1}^{\underline{b}} - \Lambda_{[\underline{a}_1 \dots \underline{a}_4} \nabla_\alpha x_{\underline{a}_5]}^{\underline{i}} + \Lambda_{[\underline{a}_1 \underline{a}_2}^{\underline{i}} \nabla_\alpha x_{\underline{a}_3 \underline{a}_4 \underline{a}_5}] - \frac{1}{2} \nabla_\alpha x_{\underline{a}_1 \dots \underline{a}_5}^{\underline{j}} \epsilon_{\underline{j}}^{\underline{i}} \Lambda + \dots \end{aligned} \quad (5.6)$$

However, it is important to remember that the local subgroup \mathcal{H} is a subgroup of $I_c(E_{11})$ and so the above parameters must be suitably restricted so that the group element h belongs to the chosen local subgroup \mathcal{H} for the brane we are considering.

5.1 The IIB string

In this subsection we will derive the equation of motion of the F1 and D1 strings in the IIB theory. The corresponding charges, Z_i^a are an $SL(2)$ doublet. We can treat the dynamics for both strings simultaneously by introducing two constants $q^i, i = 1, 2$ which are normalised so as to obey $q^2 = q^i q^i = 1$. We take the charge of the string we are considering to be given by $\langle Z_i^a \rangle = q_i Z^a$. Looking at the group element g_l of equation (5.3) we find that the dynamics of string we are considering will contain the level zero coordinate x^a , corresponding to P_a and coordinate $y_a \equiv 2q_i x_a^i$ where we have introduced a factor of 2 for reasons that will become apparent. To define the orthogonal complement to q^i we introduce $\bar{q}^i = \epsilon^{ij} q_j$. We note that $\bar{q}^i q_i = 0$ and $\bar{q}^2 = \bar{q}^i \bar{q}^i = 1$. The orthogonal level one coordinate can be taken to be $z_a \equiv 2\bar{q}_i x_a^i$ and we can write $x_a^i = \frac{1}{2} q^i y_a + \frac{1}{2} \bar{q}^i z_a$.

We must now choose the local subgroup \mathcal{H} that will lead to the string dynamics. Rather than just write it down we will now motivate our choice. Examining equation (5.6) we find that if S was in the local subalgebra \mathcal{H} it would induce a transformation with parameter Λ that interchanges the Cartan from for y_a with that for z_a . In terms of the charges it interchanges the charge $q^i Z_i^a$ with $\bar{q}^i Z_i^a$. As we want a dynamics that only contains the coordinates, or charges, associated with the chosen string we do not take the generators S to be in the local subgroup \mathcal{H} .

Examining the first of the equations in (5.6) we realise that under a transformation with parameter Λ_i^{ab} the above Cartan forms transform as

$$\delta(\nabla_\alpha x^a) = -2(q^i \Lambda_i^{ab} \nabla_\alpha y_b + \bar{q}^i \Lambda_i^{ab} \nabla_\alpha z_b)$$

and

$$\delta(\nabla_\alpha y^a) = -2q^i \Lambda_i^{ab} \nabla_\alpha x_b, \quad \delta(\nabla_\alpha z^a) = -2\bar{q}^i \Lambda_i^{ab} \nabla_\alpha x_b$$

Since the chosen string should involve the Cartan forms for the coordinates x^a and y^a and not that for z^a we should take the parameter Λ_i^{ab} to be in the form $\Lambda_i^{ab} = -q_i \tilde{\Lambda}^{ab}$ and so the generators $\hat{S}^{a_1 a_2} \equiv q_i S^{a_1 a_2 i}$ to be in \mathcal{H} and we exclude the generator $\bar{q}^i S_i^{a_1 a_2}$ from \mathcal{H} . Using similar arguments one concludes that the generator $S^{a_1 \dots a_4}$ is not in \mathcal{H} .

As a result we consider the local subgroup \mathcal{H} should contain generators that are taken from the set

$$\{J_{ab}, \hat{S}^{a_1 a_2}, \dots\} \quad (5.7)$$

We note that at this point we have not chosen the local subalgebra \mathcal{H} only stated that it is to be a subgroup of the generators listed in equation (5.7). Using the results of appendix B we find that these generators obey the commutators

$$[\hat{S}^{a_1 a_2}, \hat{S}^{b_1 b_2}] = -4\delta_{[b_1}^{[a_1} J^{a_2]}_{b_2]}, \quad [J^{a_1 a_2}, \hat{S}^{b_1 b_2}] = -4\delta_{[b_1}^{[a_1} \hat{S}^{a_2]}_{b_2]}, \quad [J^{a_1 a_2}, J^{b_1 b_2}] = -4\delta_{[b_1}^{[a_1} J^{a_2]}_{b_2]} \quad (5.8)$$

We recognise this subalgebra as the $O(10) \otimes O(10)$ as given in equation (3.6).

The transformations of the Cartan forms of equation (5.5) corresponding to generators $\hat{S}^{a_1 a_2}$ and the Lorentz transformations are given by equation (5.6) with parameter $\tilde{\Lambda}_{a_1 a_2}$ are given by

$$\delta(\nabla_\alpha x^a) = 2\nabla_\alpha y_b \tilde{\Lambda}^{ba} + 2\nabla_\alpha x^b \Lambda_b^a, \quad (5.9)$$

$$\delta(\nabla_\alpha y^a) = 2\nabla_\alpha y_b \tilde{\Lambda}^{ba} + 2\nabla_\alpha y^b \Lambda_b^a, \quad (5.10)$$

as well as

$$\delta(\nabla_\alpha z_{\underline{a}}) = 2\nabla_\alpha z_{\underline{b}} \Lambda_{\underline{a}}^{\underline{b}} + 6\nabla_\alpha x_{b_1 b_2}^{\underline{a}} \tilde{\Lambda}_{b_1 b_2}, \dots \quad (5.11)$$

$$\delta(\nabla_\alpha x_{\underline{a}_1 \underline{a}_2 \underline{a}_3}) = 6\nabla_\alpha x_{\underline{b} [\underline{a}_2 \underline{a}_3]} \Lambda_{\underline{a}_1}^{\underline{b}} + \nabla_\alpha z_{[\underline{a}_1} \tilde{\Lambda}_{\underline{a}_2 \underline{a}_3]} + \dots \quad (5.12)$$

where $+\dots$ denote higher level coordinates

Clearly, we may set $\nabla_\alpha z_{\underline{a}} = 0$ and $\nabla_\alpha x_{\underline{a}_1 \underline{a}_2 \underline{a}_3} = 0$ while preserving the symmetries of the non-linear realisation. This is the choice we now adopt.

The reader will have realised that we have arrived at exactly the same situation as we had in section four where we studied the IIA string. Indeed the generators of equation (5.7) obey the same $\text{SO}(10,10)$ algebra as those of equation (3.5) and the coordinates $x^{\underline{a}}$ and $y^{\underline{a}}$ can be identified with the coordinates of the same name in section four. The derivation of the dynamics then proceeds just as in section four with the local subalgebra \mathcal{H} being that of equation (3.8) with S_{ab} now being \hat{S}_{ab} etc. The equations of motion are equation (3.21), and finally equation (3.26).

5.2 The D3 brane

The D3 brane possess a four dimensional world volume and so we divide the indices into their longitudinal and transverse parts, in particular $\underline{a} = 0, 1, \dots, 9$ divides into $a = 0, 1, 2, 3$ and $a' = 4, 5, \dots, 9$. The generators of the vector representation are given in equation (5.2) and those of $I_c(E_{11})$ in equation (5.1). The algebra $I_c(E_{11}) \otimes_s l_1$ can be found in appendix B. The non-linear realisation of $I_c(E_{11}) \otimes_s l_1$ is constructed from the group element $g = g_l g_h$ where g_l can be found in equation (5.3).

We choose the local subgroup \mathcal{H} to be given by

$$\mathcal{H} = \{J_{ab}, J_{a'b'}, S, L_{ab'}, L_{a_1 a_2}^{i\pm}, \hat{S}, \dots\} \quad (5.13)$$

where

$$\hat{S} \equiv \epsilon^{a_1 \dots a_4} S_{a_1 \dots a_4}, \quad L_{ab'} \equiv 3! J_{ab'} + \epsilon_a^{e_1 e_2 e_3} S_{e_1 e_2 e_3 b'}, \quad L_{a_1 a_2}^{i\pm} \equiv S_{a_1 a_2}^i \pm \frac{1}{2} \epsilon^{ij} \epsilon_{a_1 a_2}^{b_1 b_2} S_{b_1 b_2 j} \quad (5.14)$$

We note that \mathcal{H} contains the $\text{SO}(2)$ generator S which is part of the $\text{SL}(2)$ symmetry of IIB theory.

We now give the algebra that the generators of \mathcal{H} obey beginning with the commutators that involve the generators S which are given by

$$[S, J_{ab}] = 0, \quad [S, J_{ab'}] = 0, \quad [S, L_{cd'}] = 0, \quad [S, \hat{S}] = 0, \quad [S, L_{a_1 a_2}^{i+}] = -\frac{1}{2} \epsilon_{ij} L_{a_1 a_2}^{j+}, \dots \quad (5.15)$$

while, those that involve the generators \hat{S} are

$$[\hat{S}, J_{ab}] = 0, \quad [\hat{S}, J_{a'b'}] = 0, \quad [\hat{S}, L_{cd'}] = -4! L_{cd'}, \quad [\hat{S}, L_{a_1 a_2}^{i+}] = 24 L_{a_1 a_2}^{i+}, \dots \quad (5.16)$$

The commutators involving the generators $L_{ab'}$ are given by

$$[L_{ab'}, L_{cd'}] = 0, \quad [L_{a_1 a_2}^{i+}, L_{cd'}] = 0, \quad \dots \quad (5.17)$$

The commutators involving the generators $L_{a_1 a_2}^{i+}$ are given by

$$[L_{a_1 a_2}^{i+}, L_{b_1 b_2}^{j+}] = 0, \dots \quad (5.18)$$

As noted in section two, equation (2.12) we can use the local symmetry to choose the parts of g_h that are in \mathcal{H} to vanish. Indeed we can choose the generators which are in $I_c(E_{11})$ but not in \mathcal{H} , up to level two, to be $J^a{}_{b'}$, $L_{a_1 a_2}^{i-}$, $S_{ab'}^i$ and $S_{a'b'}^i$ and as a result we may choose g_h to be of the form

$$g_h = e^{-\varphi_a{}^{b'} J^a{}_{b'}} e^{\varphi^{i+a_1 a_2} L_{a_1 a_2}^{i-}} e^{S_{ab'}^i \varphi_i^{ab'}} e^{S_{a'b'}^i \varphi_i^{a'b'}} \dots \quad (5.19)$$

where

$$\varphi_{\pm a_1 a_2 i} \equiv \frac{1}{2}(\varphi_{a_1 a_2}^i \pm \frac{1}{2}\epsilon^{ij}\epsilon_{a_1 a_2}{}^{b_1 b_2}\varphi_{b_1 b_2 j}) = \pm \frac{1}{2}\epsilon^{ij}\epsilon_{a_1 a_2}{}^{b_1 b_2}\varphi_{b_1 b_2 j} \quad (5.20)$$

We note that $T^{i+a_1 a_2} R_{a_1 a_2}^{i-} = 0$ for any two objects $T^{i+a_1 a_2}$ and $R_{a_1 a_2}^{i-}$ where the \pm projections are as expected.

We now choose an alternative basis of the vector representation to that given in equation (5.2) that is suited to the action of the local subgroup \mathcal{H} . We take the basis

$$N_a^\pm, P_{a'}, Z_a^i, N_{a'}^{i\pm}, Z^{a_1 a_2 b'}, Z^{ab'_1 b'_2}, Z^{a'_1 a'_2 a'_3}, \dots \quad (5.21)$$

where

$$N_a^\pm = P_a \pm \frac{1}{2.3!}\epsilon_{ae_1 e_2 e_3} Z^{e_1 e_2 e_3}, N_{a'}^{i\pm} = Z_{a'}^i \mp \frac{1}{4!}\epsilon^{e_1 \dots e_4} Z_{e_1 \dots e_4 a'}^i \quad (5.22)$$

We now present the commutators of the vector representation with the generators of \mathcal{H} . The action of the Lorentz generators is standard and so we begin with the commutators containing S :

$$\begin{aligned} [S, P_{a'}] &= 0, \quad [S, N_a^\pm] = 0, \quad [S, Z^{a_1 a_2 b'}] = 0, \quad [S, Z^{ab'_1 b'_2}] = 0, \\ [S, N_{a'}^{i\pm}] &= -\frac{1}{2}\epsilon^{ij} N_{a'}^{\pm j}, \quad [S, Z_c^i] = -\frac{1}{2}\epsilon^{ij} Z_c^j \end{aligned} \quad (5.23)$$

The commutators with the generators $L_{ab'}$ are given by

$$\begin{aligned} [L_{ab'}, N_c^-] &= 0, \quad [L_{ab'}, N_c^+] = -2.3!\eta_{ac} P_{b'} + 3\epsilon_{ace_1 e_2} Z^{e_1 e_2 b'} \\ [L_{ab'}, P_{c'}] &= 3!\eta_{b'c'} N_a^-, \quad [L_{ab'}, Z_c^i] = -3!\eta_{ac} N_i^{-b'} \\ [L_{ab'}, N_{ic'}^-] &= 0, \quad [L_{ab'}, N_{ic'}^+] = 2.3!\eta_{b'c'} Z_a^i - 2\epsilon_{ae_1 e_2 e_3} Z_i^{e_1 e_2 e_3 b' c'}, \\ [L_{ab'}, Z_{c_1 c_2 c_3}] &= -3.3!\eta_{a[c_1|} Z_{b'|c_2 c_3]} + 12\epsilon_{ac_1 c_2 c_3} P_{b'}, \\ [L_{ab'}, Z_{c_1 c_2 d'}] &= 12\eta_{a[c_1|} Z_{|c_2] b' d'} - 12\eta_{b' d'} \epsilon_{ac_1 c_2 e} N_e^-, \\ [L_{ab'}, Z_{c'_1 c'_2 c'_3}] &= 18\eta_{b'[c'_1} Z_{c'_2 c'_3] a}, \quad [L_{ab'}, Z^{cd'_1 d'_2}] = -3!\eta_{ac} Z_{b' d'_1 d'_2} - 12\eta_{b' [d'_1} Z_{d'_2] ac} \end{aligned} \quad (5.24)$$

The commutators of $L_{a_1 a_2}^{i+}$ with the vector representation generators are given by

$$\begin{aligned}
[L_{a_1 a_2}^{i+}, N_b^-] &= 0, \quad [L_{a_1 a_2}^{i+}, N_b^+] = \varepsilon^{ij} \varepsilon_{b a_1 a_2 c} Z_c^j + 2\eta_{b[a_1} Z_{a_2]}^i \\
[L_{a_1 a_2}^{i+}, P_{b'}] &= 0, \quad [L_{a_1 a_2}^{i+}, Z_b^j] = 2\varepsilon^{ij} \varepsilon^{a_1 a_2 b d} N_d^- + 4\delta^{ij} \eta_{b[a_1} N_{a_2]}^-, \\
[L_{a_1 a_2}^{i+}, Z_{b'}^j] &= -\varepsilon^{ij} Z^{a_1 a_2 b'} - \frac{1}{2} \delta^{i,j} \varepsilon_{a_1 a_2 c_1 c_2} Z^{c_1 c_2 b'}, \\
[L_{a_1 a_2}^{i+}, N_{b'}^{-j}] &= 0, \quad [L_{a_1 a_2}^{i+}, N_{b'}^{+j}] = -\delta^{i,j} \varepsilon_{a_1 a_2 c_1 c_2} Z^{c_1 c_2 b'} - 2\varepsilon^{ij} Z^{a_1 a_2 b'} \\
[L_{a_1 a_2}^{i+}, Z^{b_1 b_2 b_3}] &= 6\varepsilon^{ij} \delta_{a_1 a_2}^{[b_1 b_2} Z_j^{b_3]} - 3\delta_j^i \varepsilon_{a_1 a_2 [b_1 b_2} Z_{j|b_3]} \\
[L_{a_1 a_2}^{i+}, Z^{b_1 b_2 c'}] &= -\varepsilon_{a_1 a_2 b_1 b_2} N_{c'}^{i-} + 2\varepsilon^{ij} \delta_{a_1 a_2}^{b_1 b_2} N_j^{-c'}, \\
[L_{a_1 a_2}^{i+}, N_-^{cd_1 d_2'}] &= 2Z^{i a_1 a_2 c d_1' d_2'} + \varepsilon^{ij} \varepsilon_{a_1 a_2 b_1 b_2} Z_j^{b_1 b_2 c d_1' d_2'}, \quad [L_{a_1 a_2}^{i+}, N_+^{cd_1 d_2'}] = 0 \\
[L_{a_1 a_2}^{i+}, Z^{c_1' c_2' c_3'}] &= Z^{i a_1 a_2 c_1' c_2' c_3'} + \frac{1}{2} \varepsilon^{ij} \varepsilon_{a_1 a_2 b_1 b_2} Z_j^{b_1 b_2 c_1' c_2' c_3'}, \tag{5.25}
\end{aligned}$$

Taking the commutators with $\hat{S} = \varepsilon^{a_1 \dots a_4} S_{a_1 \dots a_4}$, we find the following

$$\begin{aligned}
[\hat{S}, P_{a'}] &= 0, \quad [\hat{S}, N_a^\pm] = \pm 4.3! N_b^\pm, \quad [\hat{S}, N_{b'}^{\pm i}] = \pm 4! N_i^{\pm b'}, \quad [\hat{S}, Z_c^i] = 0, \\
[\hat{S}, N_\pm^{c_1 c_2 d'}] &= \pm N_\pm^{cd_1 d_2'}, \quad [\hat{S}, Z^{b_1 b_2 c'}] = 0, \quad [\hat{S}, Z^{bc_1 c_2}] = 0, \quad [\hat{S}, Z^{c_1' c_2' c_3'}] = 0 \tag{5.26}
\end{aligned}$$

Examining the above commutators we find that

$$N_c^-, N_{ic'}^-, \dots$$

is an irreducible representation under the local subalgebra \mathcal{H} .

When written in terms of the basis of equation (5.21) the Cartan forms are given by

$$\mathcal{V} = \mathcal{E}^{\pm a} N_a^\pm + \mathcal{E}^{c'} P_{c'} + N_{c'}^{\pm i} \mathcal{E}^{\pm c'}{}_i + \mathcal{E}_i^c Z_c^i + \mathcal{E}_{a_1 a_2 b'} Z^{a_1 a_2 b'} + \mathcal{E}_{ab_1 b_2'} Z^{ab_1 b_2'} + \mathcal{E}_{a_1' a_2' a_3'} Z^{a_1' a_2' a_3'} + \dots \tag{5.27}$$

where the \pm are summed over and

$$\begin{aligned}
\mathcal{E}^{\pm a} &= \frac{1}{2} \nabla x^a \mp \varepsilon^{ab_1 b_2 b_3} \nabla x_{b_1 b_2 b_3}, \quad \mathcal{E}^{a'} = \nabla x^{a'}, \quad \mathcal{E}_a^i = \nabla x_a^i, \\
\mathcal{E}^{\pm a' i} &= \frac{1}{2} (\nabla x_i^{a'} \pm 5\varepsilon_{e_1 \dots e_4} x^{e_1 \dots e_4 a'}{}_i), \\
\mathcal{E}^{a_1 a_2 b'} &= \nabla x^{a_1 a_2 b'}, \quad \mathcal{E}^{cd_1 d_2'} = \nabla x^{cd_1 d_2'}, \quad \mathcal{E}^{d_1' d_2' d_3'} = \nabla x^{d_1' d_2' d_3'} \tag{5.28}
\end{aligned}$$

In these equations the \mathcal{E} 's are forms, that is, $\mathcal{E}^\bullet = d\xi^\alpha \mathcal{E}_\alpha^\bullet$ and similarly $\nabla = d\xi^\alpha \nabla_\alpha$. The variation of the Cartan forms under the local \mathcal{H} transformations is given by

$$\delta \mathcal{V} = [\hat{\Lambda} \hat{S} + \Lambda S + \Lambda^{ab'} L_{ab'} + \Lambda^{-a_1 a_2 i} L_{+a_1 a_2 i}, \mathcal{V}] \tag{5.29}$$

and we find that

$$\delta\mathcal{E}^+{}_a = 4!\mathcal{E}^+{}_a\hat{\Lambda}, \quad \delta\mathcal{E}_{b'} = -12\mathcal{E}^+{}_a\Lambda^a{}_{b'}, \quad (5.30)$$

$$\delta\mathcal{E}_a{}^i = \frac{\Lambda}{2}\varepsilon^{ij}\mathcal{E}^a{}_j + 12\Lambda^{ab'}\mathcal{E}^+{}_{b'}{}^i + 4\mathcal{E}^+_c\Lambda^{-c}{}_{ai}, \quad \delta\mathcal{E}^{+i}{}_{a'} = \frac{\Lambda}{2}\mathcal{E}^+{}_{a'j}\varepsilon^{ij} + 4\hat{\Lambda}\mathcal{E}^+{}_{a'}{}^i \quad (5.31)$$

$$\delta\mathcal{E}_{a_1a_2b'} = 3\Lambda^{db'}\varepsilon_{dca_1a_2}\mathcal{E}^+_c + 12\mathcal{E}_{[a_1|d'b'}\Lambda_{|a_2]}{}^{d'} - 4\varepsilon^{ij}\Lambda_{-a_1a_2i}\mathcal{E}^+{}_{b'j} \\ \delta\mathcal{E}_{a'_1a'_2a'_3} = -3!\Lambda_{e[a'_1]}\mathcal{E}^\pm{}_{e|a'_2a'_3]}, \quad \delta\mathcal{E}^{ab'_1b'_2} = -12\varepsilon^{da[b'_1]}\Lambda^{d|b'_2]} + 18\mathcal{E}^{d'b'_1b'_2}\Lambda^a{}_{d'} \quad (5.32)$$

$$\delta\mathcal{E}^-{}_a = -4!\mathcal{E}^-{}_a\hat{\Lambda} + 3!\mathcal{E}_{b'}\Lambda^{ab'} - 12\varepsilon_{dc_1c_2a}\Lambda^{ab'}\mathcal{E}^{c_1c_2b'} + 8\mathcal{E}^{+bi}\Lambda_{-bai} \\ \delta\mathcal{E}^{-i}{}_{a'} = -3!\mathcal{E}_d{}^i\Lambda^{da'} - 4\hat{\Lambda}\mathcal{E}^{-a'}{}^i + \frac{\Lambda}{2}\mathcal{E}^{-a'}{}_{aj}\varepsilon^{ij} - 4\varepsilon^{ij}\Lambda_{-c_1c_2}{}^j\mathcal{E}^{c_1c_2a'} \quad (5.33)$$

Examining equation (5.30) we find that we can consistently set to zero

$$\mathcal{E}_\alpha{}^{a'} = \nabla_\alpha x^{a'} = 0, \quad \mathcal{E}_\alpha{}^{+a} = \frac{1}{2}\nabla_\alpha x^a - \varepsilon^{ab_1b_2b_3}\nabla_\alpha x^{b_1b_2b_3} = 0 \quad (5.34)$$

that is, we can set these constraints and those of equation (5.33) to zero while preserving all the symmetries of the non-linear realisation.

We now restrict the discussion to the **linearised** theory. Examining the above local variations we see that we can set

$$\mathcal{E}^L{}_{[\alpha a]}{}^i = 0, \quad (5.35)$$

as well as

$$\mathcal{E}_\alpha{}^{+a'i} = 0 \quad (5.36)$$

and preserve all the symmetries of the non-linear realisation.

To find the meaning of the above conditions for the linearised theory we compute their expressions in terms of the fields. As a first step we calculate the Cartan forms for a general group element, g_h , that is, one that belongs to $I_c(E_{11})$ and so has the form

$$g_h = e^{-J_{a_1a_2}\phi^{a_1a_2} + S_i^{a_1a_2}\phi^{i}_{a_1a_2} + S^{a_1\cdots a_4}\phi_{a_1\cdots a_4} + \phi S} \quad (5.37)$$

Proceeding in this way we have not yet used the local subalgebra \mathcal{H} to restrict the group element ϕ . The result at the linearised level is given by

$$\nabla_\alpha x^a = \partial_\alpha x^a - 4\phi_i^{ab}\partial_\alpha x^i_{\underline{b}} + 2\partial_\alpha x^b\phi_{\underline{b}}{}^a + 48\partial_\alpha x^{b_1b_2b_3}\phi_{b_1b_2b_3}{}^a + \dots \quad (5.38)$$

$$\nabla_\alpha x^i_{\underline{a}} = \partial_\alpha x^i_{\underline{a}} - \phi_i^{ab}\partial_\alpha x^b + 2\partial_\alpha x^i_{\underline{b}}\phi_{\underline{a}}{}^b - 6\partial_\alpha x^{b_1b_2a}\epsilon_{ij}\phi^j_{b_1b_2} + 120\partial_\alpha x^{b_1\cdots b_4a}\phi^{b_1\cdots b_4} - \frac{1}{2}\partial_\alpha x^j_{\underline{a}}\epsilon_j{}^i\phi + \dots \quad (5.37)$$

$$\nabla_\alpha x_{a_1a_2a_3} = \partial_\alpha x_{a_1a_2a_3} + 6\partial_\alpha x_{b[a_2a_3}]\phi^b_{a_1]} + 2\partial_\alpha x^b\phi_{ba_1a_2a_3} - 20\partial_\alpha x^i_{b_1b_2a_1a_2a_3}\phi^{b_1b_2}_i - \partial_\alpha x^j_{[a_1}\epsilon_{ij}\phi^i_{a_2a_3]} + \dots \quad (5.38)$$

$$\nabla_\alpha x^i_{\underline{a_1}\cdots\underline{a_5}} = \partial_\alpha x^i_{\underline{a_1}\cdots\underline{a_5}} + 10\partial_\alpha x^i_{\underline{b}[a_2\cdots a_5}]\phi^b_{a_1]} - \phi_{[a_1\cdots a_4}\partial_\alpha x^i_{a_5]} + \phi^i_{[a_1a_2}\partial_\alpha x_{a_3a_4a_5]} - \frac{1}{2}\partial_\alpha x^j_{\underline{a_1}\cdots\underline{a_5}}\epsilon_j{}^i\phi + \dots \quad (5.39)$$

where $+\dots$ means terms that are higher level in the fields ϕ and x .

If we choose static gauge $\partial_\alpha x^a = \delta_\alpha^a$ then as zeroth order we conclude from equation (5.34) that $\partial_\alpha x_{a_1 a_2 a_3} = -\frac{1}{2 \cdot 3!} \epsilon_{\alpha a_1 a_2 a_3}$. Whereupon we find that to at most either first order in the fields x or ϕ that

$$\nabla_\alpha x^{a'} = \partial_\alpha x^{a'} + 2\phi_\alpha^{a'} - 4\epsilon_{\alpha e_1 e_2 e_3} \phi^{e_1 e_2 e_3 a'} + \dots \quad (5.40)$$

$$\nabla_\alpha x_{a_1 a_2 b'} = \partial_\alpha x_{a_1 a_2 b'} + 2\phi_{\alpha a_1 a_2 b'} - \frac{1}{6} \epsilon_{\alpha a_1 a_2 e} \phi^e_{b'} + \dots \quad (5.41)$$

We observe that

$$\nabla_\alpha x^{a'} + 2\epsilon_\alpha^{\beta e_1 e_2} \nabla_\beta x_{e_1 e_2}^{a'} = \partial_\alpha x^{a'} + 2\epsilon_\alpha^{\beta e_1 e_2} \partial_\beta x_{e_1 e_2}^{a'} \quad (5.42)$$

while

$$\nabla_\alpha x^{a'} - 2\epsilon_\alpha^{\beta e_1 e_2} \nabla_\beta x_{e_1 e_2}^{a'} = \partial_\alpha x^{a'} - 2\epsilon_\alpha^{\beta e_1 e_2} \partial_\beta x_{e_1 e_2}^{a'} + 2(\phi_\alpha^{a'} - 4\epsilon_{\alpha e_1 e_2 e_3} \phi^{e_1 e_2 e_3 a'}) \quad (5.43)$$

The constraint $\mathcal{E}_\alpha^{a'} = 0$ of equation (5.34) and $\mathcal{E}_\alpha^{+a_1 a_2 b'} = 0$ of equation (5.35) imply that the objects in equations (5.42) and (5.43) vanish. Implementing this we see that equation (5.43) allows us to solve for the combination of ∂x 's that occurs in terms of the fields ϕ 's and is an inverse Higgs condition while equation (5.42) is an equation of motion. Indeed taking a derivative it is the correct equation of motion for the D3 brane at the linearised level.

As explained in the context of the other branes our choice of local subalgebra is so as to ensure that only the combination of ϕ 's that occurs in equation (3.43) appears in the group element g_h once we have used the local subalgebra to restrict the ϕ 's in g_h .

We now investigate the Cartan form that contains the world volume vector x_a^i in the linearised theory, for terms at most linear in either x and ϕ 's it takes the form

$$\nabla_{[\alpha} x_{a]}^i = \partial_{[\alpha} x_{a]}^i + 4\phi_{+\alpha a}^i \quad (5.44)$$

where $\phi_{\pm\alpha a}^i = \frac{1}{2}(\phi_{\alpha a}^i \pm \frac{1}{2}\epsilon_{\alpha a e_1 e_2} \epsilon^{ij} \phi^{e_1 e_2 j})$. Using the constraint $\mathcal{E}_{[\alpha a]}^i = 0$ of equation (5.35) we find that

$$\partial_{[\alpha} x_{a]}^i - \frac{1}{2} \epsilon_{\alpha a}^{\beta e} \epsilon^{ij} \partial_\beta x_{e j} = 0 \quad (5.45)$$

and

$$\partial_{[\alpha} x_{a]}^i + \frac{1}{2} \epsilon_{\alpha a}^{\beta e} \epsilon^{ij} \partial_\beta x_{e j} + 8\phi_{+\alpha a}^i = 0 \quad (5.45)$$

The last equations just express the fact that the anti-self-dual part of the field strength $f_{ab}^i = \partial_a x_b^i - \partial_b x_a^i$ is expressed in terms of the field $\phi_{+\alpha a}^i$ while equation (5.45) implies that the self-dual part of the field strength vanishes. This leads to the correct linearised equation for the vector field of the D3 brane.

We now consider the **non-linear** theory using similar arguments we used for the M5 brane. The constraints of equation (5.34) can be written, using section six, in the form

$$\sqrt{-\gamma} \gamma^{\alpha\beta} \nabla_\beta x^a = 2\epsilon^{\alpha\beta\gamma_1\gamma_2} \nabla_\beta x^{a b_1 b_2} \nabla_{\gamma_1} x_{b_1} \nabla_{\gamma_2} x_{b_2} \quad (5.46)$$

We also adopt the constraint of equation (5.36) however, we must modify that of equation (5.35) as it is not reparameterisation invariant. For the non-linear theory we must take the D3 brane to have an enlarged world volume. We take it to have the coordinates $\xi^\alpha = \{\xi^\alpha, \xi_\alpha^j\}$, in other words, the brane moves in some of the directions with higher level coordinates. We define

$$E_{\underline{\alpha}}^A = \begin{pmatrix} \nabla_\alpha x^a & \nabla_\alpha x_b^j \\ \nabla_i^a x^a & \nabla_i^a x_b^j \end{pmatrix} \quad (5.47)$$

We note that

$$\delta(\nabla x^a) = -4\Lambda_-^{ab} \nabla_i x_b^i = -4\Lambda_-^{ab} E_b^i, \quad \delta(\nabla x_a^i) = 0 \quad (5.48)$$

under the transformations with parameters Λ_-^{ab} and Λ_{ab} provided we use the constraints of equations (5.34) and (5.36). Instead of equation (5.35) we adopt the condition

$$\mathcal{E}_{a_1 a_2}^i \equiv \nabla_{[a_1} x_{a_2]}^i - \nabla_b x_{[a_1]}^j \nabla_j^b x_{|a_2]}^i = 0 \quad (5.49)$$

where $\nabla_a = (s^{-1})_a^\alpha \nabla_\alpha$, $s_\alpha^a = \nabla_\alpha x^a$ and $\nabla_j^b = (E^{-1})_j^{\underline{b}} \nabla_{\underline{a}}$. Using the variations of equation (5.48) and one finds that equation (5.49) is invariant under the local transformations provided one uses the on-shell condition (5.49) and neglects terms which contain higher level derivatives. As we mentioned for the M5 brane we are finding equations of motion to lowest order in the usual derivatives, that is, they just contain derivatives with respect to the usual world volume coordinates ξ^α . In doing this we must add terms in the equation being varied which contain derivatives with respect to the coordinates ξ_α^j . In view of the limited nature of the calculations for the non-linear theory the proposal for the non-linear theory given here must be regarded as suggestion rather than a proven result.

Neglecting all the terms with higher level derivatives the equations for the non-linear theory are those of equation (5.46) and the equation of motion for the gauge field is given by

$$\nabla_{[a_1} x_{a_2]}^i = \frac{1}{2} \epsilon_{a_1 a_2}^{b_1 b_2} \epsilon^{ij} \nabla_{b_1} x_{b_2 j} \quad (5.50)$$

6 Branes in seven dimensions

The seven dimensional theory emerges when we decompose E_{11} into $GL(7) \otimes Sl(5)$ which is the algebra that emerges when we delete node seven in the E_{11} Dynkin diagram. The generators in this decomposition at low level are easily computed using the Nutma programme SimpLie [30] and are given by

$$\begin{aligned} K_{\underline{b}}^a, \quad R_N^M; \quad R^{aMN}; \quad R^{a_1 a_2}_M; \quad R^{a_1 a_2 a_3}_M; \quad R^{a_1 \dots a_4}_{MN}; \quad R^{a_1 \dots a_5}_M{}^N, \\ R^{a_1 \dots a_4, \underline{b}}, \quad R^{a_1 \dots a_6}_{MN, P}; \quad R^{a_1 \dots a_6 (MN)}, \quad R^{a_1 \dots a_5, \underline{b}}{}^{MN}, \dots \end{aligned} \quad (6.1.1)$$

where $a, b, \dots = 0, 1, \dots, 6$ and $M, N, \dots = 1, 2, 3, 4, 5$. The generators in $I_c(E_{11})$ algebra are

$$\begin{aligned} J_{\underline{b}}^a = K_{\underline{b}}^a - K_{\underline{a}}^b, \quad S_N^M = R_N^M - R^N_M, \quad S^{aMN} = R^{aMN} - R_{aMN}, \\ S^{a_1 a_2}_M = R^{a_1 a_2}_M + R_{a_1 a_2}^M, \quad S^{a_1 a_2 a_3}_M = R^{a_1 a_2 a_3}_M - R_{a_1 a_2 a_3 M}, \dots \end{aligned} \quad (6.1.2)$$

While the elements of the vector representation are

$$P_{\underline{a}}; \quad Z^{MN}; \quad Z^a_M; \quad Z^{a_1 a_2 M}; \quad Z^{a_1 a_2 a_3}_{MN}; \quad Z^{a_1 a_2 a_3, \underline{b}}; \quad Z^{a_1 \dots a_4}; \quad Z^{a_1 \dots a_4 M}_N, \\ Z^{a_1 \dots a_5 MN}, \quad Z^{a_1 \dots a_5 (MN)}, \quad Z^{a_1 \dots a_5}_{MN, P}, \quad Z^{a_1 \dots a_4, \underline{b} MN}, \dots \quad (6.1.3)$$

The $E_{11} \otimes_s l_1$ algebra in this decomposition has been worked out at low levels by Michaela Pettit and the author and will be given elsewhere [33].

The non-linear realisation $I_c(E_{11}) \otimes_s l_1$ is constructed from $g = g_L g_h$ where

$$g_l = \exp(x^a P_{\underline{a}} + x_{MN} Z^{MN} + x_{\underline{a}}^M Z^a_M + x_{a_1 a_2 M} Z^{a_1 a_2 M} + x_{a_1 a_2 a_3}^{MN} Z^{a_1 a_2 a_3}_{MN} \\ + x_{a_1 a_2 a_3, \underline{b}} Z^{a_1 a_2 a_3, \underline{b}} + x_{a_1 \dots a_4} Z^{a_1 \dots a_4} + x_{a_1 \dots a_4 M}^N Z^{a_1 \dots a_4 M}_N + \dots) \quad (6.1.4)$$

We will initially consider the most general group element $g_h \in I_c(E_{11})$, that is, before we have used the local subalgebra \mathcal{H} to set some of the fields in g_h to zero. Thus we take

$$g_h = \exp(\phi_{\underline{a}}^{\underline{b}} J^a_{\underline{b}} + \phi_M^N S^M_N + \phi_{\underline{a} MN} S^{a MN} + \phi_{a_1 a_2}^M S^{a_1 a_2}_M + \phi_{a_1 a_2 a_3 M} S^{a_1 a_2 a_3 M} + \dots) \quad (6.1.5)$$

The Cartan forms which belong to the vector representation of the E_{11} algebra can be written in the form

$$\mathcal{V}_l = \nabla x^a P_{\underline{a}} + \nabla x_{PQ} Z^{PQ} + \nabla x_{\underline{a}}^M Z^a_M + \nabla x_{a_1 a_2 M} Z^{a_1 a_2 M} + \nabla x_{a_1 a_2 a_3}^{MN} Z^{a_1 a_2 a_3}_{MN} \\ + \nabla x_{a_1 a_2 a_3, \underline{b}} Z^{a_1 a_2 a_3, \underline{b}} + \nabla x_{a_1 \dots a_4} Z^{a_1 \dots a_4} + \nabla x_{a_1 \dots a_4 M}^N Z^{a_1 \dots a_4 M}_N + \dots \quad (6.1.6)$$

Using equation (2.9), the group element of equation (6.1.5) and the results of reference [33], we find that the Cartan forms are given at low levels by

$$\nabla x^a = \partial x^a + 2\partial x^{\underline{b}} \phi_{\underline{b}}^a + 2\partial x_{MN} \phi^{a MN} - 2\partial x^{\underline{b} S} \phi_{\underline{b} a S} - 12\phi_{\underline{a} \underline{b}_1 \underline{b}_2 M} \partial x^{\underline{b}_1 \underline{b}_2 M} + \dots, \\ \nabla x_{MN} = \partial x_{MN} - 2\partial x_{S[M} \phi^S_{N]} - \partial x^{\underline{b}} \phi_{\underline{b} MN} - \frac{1}{2} \partial x^a P_{\underline{a}} \phi_{\underline{a}}^{RS} \epsilon_{RSPMN} + 2\phi_{a_1 a_2 [M} \partial x^{a_1 a_2}_{N]} + \dots, \\ \nabla x_{\underline{a}}^P = \partial x_{\underline{a}}^P + 2\partial x_{\underline{b}}^P \phi_{\underline{a}}^{\underline{b}} + \partial x_a^Q \phi_Q^P + \phi_{\underline{a}}^{MN} \partial x^{RQ} \epsilon_{MNRQP} + 4\phi_{\underline{b} MP} \partial x_{\underline{a}}^{\underline{b} M} + 2\phi_{\underline{a} \underline{b} P} \partial x^{\underline{b}} + \dots \\ \nabla x_{a_1 a_2 P} = \partial x_{a_1 a_2 P} + 4\partial x^{\underline{b}}_{[a_2 | P} \phi_{\underline{b} | a_1]} + \partial x_{a_1 a_2 Q} \phi_Q^P - 2\phi_{[a_1 | Q P} \partial x_{a_2]}^Q - 2\phi_{a_1 a_2 R} \partial x^R_P \\ + 6\phi_{\underline{b} a_1 a_2 P} \partial x^{\underline{b}} + \frac{3}{2} \phi_{\underline{b} a_1 a_2}^{MN} \partial x_{\underline{b} a_1 a_2}^{RS} \epsilon_{MNRSP} + \dots \\ \nabla x_{a_1 a_2 a_3}^{PQ} = \partial x_{a_1 a_2 a_3}^{PQ} + 6\partial x_{\underline{b} [a_1 a_2]}^{PQ} \phi_{\underline{b} | a_3]}^{\underline{b}} + 2\partial x_{a_1 a_2 a_3 S}^{[Q |} \phi^{S | P]} - \phi_{[a_1 | MN} \partial x_{a_2 a_3] T} \epsilon^{MNT PQ} \\ - 2\phi_{[a_1 a_2}^{[P} \partial x_{a_1]}^Q] + 2\phi_{a_1 a_2 a_3 M} \partial x_{RS} \epsilon^{MRSPQ} - 8\phi_{\underline{b} a_1 a_2 a_3}^{PQ} \partial x^{\underline{b}} + \dots \quad (6.1.7)$$

where we are using form notation for the derivatives. These expressions applies to all branes in seven dimensions once we restrict the ϕ fields using the local subalgebra \mathcal{H} corresponding to the brane being considered.

Under a local transformation $I_c(E_{11})$ the Cartan forms transform as follows

$$\begin{aligned}
\delta(\nabla x^{\underline{a}}) &= 2\nabla x^{\underline{b}}\Lambda_{\underline{b}}^{\underline{a}} + 2\nabla x_{MN}\Lambda^{aMN} - 2\nabla x^{\underline{b}S}\Lambda_{\underline{b}aS} - 12\Lambda_{\underline{a}b_1b_2M}\nabla x^{b_1b_2M}, \\
\delta(\nabla x_{MN}) &= -2\nabla x_{S[M}\Lambda^S_{N]} - \nabla x^{\underline{b}}\Lambda_{\underline{b}MN} - \frac{1}{2}\nabla x^{\underline{a}P}\Lambda_{\underline{a}}^{RS}\epsilon_{RSPMN} + 2\Lambda_{\underline{a}_1\underline{a}_2[M}\nabla x^{\underline{a}_1\underline{a}_2N]}, \\
\delta(\nabla x_{\underline{a}}^P) &= 2\nabla x_{\underline{b}}^P\Lambda_{\underline{a}}^{\underline{b}} + \nabla x_a^Q\Lambda_Q^P + \Lambda_{\underline{a}}^{MN}\nabla x^{RQ}\epsilon_{MNRQP} + 4\Lambda_{\underline{b}MP}\nabla x_{\underline{a}}^{\underline{b}M} + 2\Lambda_{\underline{a}bP}\nabla x^{\underline{b}} \\
\delta(\nabla x_{\underline{a}_1\underline{a}_2}P) &= 4\nabla x_{[\underline{a}_2|P}\Lambda_{\underline{b}|\underline{a}_1]}^{\underline{b}} + \nabla x_{\underline{a}_1\underline{a}_2}Q\Lambda_Q^P - 2\Lambda_{[\underline{a}_1|Q}P\nabla x_{|\underline{a}_2]}^Q - 2\Lambda_{\underline{a}_1\underline{a}_2R}\nabla x^R_P \\
&\quad + 6\Lambda_{\underline{b}\underline{a}_1\underline{a}_2P}\nabla x^{\underline{b}} + \frac{3}{2}\Lambda^{\underline{b}MN}\nabla x_{\underline{b}\underline{a}_1\underline{a}_2}^{RS}\epsilon_{MNRSP} \\
\delta(\nabla x_{\underline{a}_1\underline{a}_2\underline{a}_3}^{PQ}) &= 6\nabla x_{\underline{b}[\underline{a}_1\underline{a}_2]}^{PQ}\Lambda_{\underline{a}_3]}^{\underline{b}} + 2\nabla x_{\underline{a}_1\underline{a}_2\underline{a}_3}S^{[Q}|\Lambda^{S|P]} - \Lambda_{[\underline{a}_1|MN}\nabla x_{|\underline{a}_2\underline{a}_3]T}\epsilon^{MNT PQ} \\
&\quad - 2\Lambda_{[\underline{a}_1\underline{a}_2}^{[P}\nabla x_{\underline{a}_3]}^{Q]} + 2\Lambda_{\underline{a}_1\underline{a}_2\underline{a}_3M}\nabla x_{RS}\epsilon^{MRSPQ} - 8\Lambda_{\underline{b}\underline{a}_1\underline{a}_2\underline{a}_3}^{PQ}\nabla x^{\underline{b}} \quad (6.1.8)
\end{aligned}$$

The above transformations belong to $I_c(E_{11})$, but the actual local transformations belong to the subalgebra \mathcal{H} and they can be obtained from the above by restricting the parameters Λ . This will provide a quick way of finding the local transformations for the different branes. The reader will notice that equations (6.1.7) and (6.1.8) are closely related as becomes clear once one examines equations (2.8) and (2.15).

6.2 The one brane

The one brane has a two dimensional world volume and so we take $a, b, \dots = 0, 1$ and $a', b', \dots = 2, \dots, 6$. The one brane charge $< Z_M^a >$ can be chosen to be of the form $< Z_M^a > = q_M Z^a$ where $q_M q_M = 1$. The different choices of the parameter q_M allow us to turn on different charges using the same formalism. No matter what choice of q_M we take it will break the $SO(5)$ symmetry to $SO(4)$ which will belong to the local subalgebra \mathcal{H} . We also introduce an orthogonal set of parameters q_M^i , $i = 1, 2, 3, 4$ which obey $q_M^i q_M^j = \delta^{ij}$ and $q_M q_M^i = 0$. Using these we can write the preserved $SO(4)$ generators as $S^i_j = J^M_N q_M^i q_j^N$ where we use summation convention on the $SO(5)$ indices.

Examining the group element g_l of equation (6.1.4) we conclude that the coordinate corresponding to the active charge is $y_a \equiv x_a^M q_M$. We introduce the orthogonal coordinates as follows

$$y_a \equiv x_a^M q_M, \quad y_a^i = x_a^M q_M^i, \quad \text{or equivalently} \quad x_a^M = q^M y_a + q_i^M y_a^i \quad (6.2.1)$$

The coordinates x_a^M belong to the 5 of $SO(5)$ and the above decomposes them into the $4 + 1$ of $SO(4)$. The coordinates x_{MN} belong to the 10 of $SO(5)$ which decomposes into the $6 + 4$ of $SO(4)$ as follows

$$x^{ij} = q_M^i q_N^j x^{MN}, \quad x^i = q_M^i q_N x^{MN}, \quad \text{or equivalently,} \quad x_{MN} = q_M^i q_N^j x_{ij} + (q_M^i q_N - q_N^i q_M) x_i \quad (6.2.2)$$

We adopt analogous decompositions for all the objects in the 5 and 10 representations of $SO(5)$, for example

$$S^{a_1 a_2 M} = q^M S^{a_1 a_2} + q_i^M S^{a_1 a_2 i} \quad (6.2.3)$$

We will choose our local subalgebra \mathcal{H} to contain the generators

$$\mathcal{H} = \{J_{ab}, J_{a'b'}, S^i_J, L_{ab'}, S_{-aij}, S, S_{a'ij}, \dots\} \quad (6.2.4)$$

where

$$L_{ab'} = J_{ab'} + \epsilon_a^c S_{cb'}, S = \frac{1}{2} \epsilon_{a_1 a_2} S^{a_1 a_2} \quad (6.2.5)$$

$$S_{\pm aij} = \frac{1}{2} (S_{aij} \pm \epsilon_{ab} \epsilon_{ijkl} S^{bkl}) \quad (6.2.6)$$

The generators of the local subalgebra \mathcal{H} obey the commutators

$$\begin{aligned} [L_{ab'}, L_{cd'}] &= 0, [S_{-aij}, S_{-bkl}] = 0, [S, L_{ab'}] = -L_{ab'}, [S, S_{-aij}] = -S_{-aij}, \\ [L_{ab'}, S_{-bij}] &= 0, \dots \end{aligned} \quad (6.2.7)$$

as well as the usual commutators with the Lorentz generators J_{ab} and $J_{a'b'}$. One finds that the commutator of S_{-aij} and S^{bk} generates S^l which is not in \mathcal{H} and so we may conclude that S^{bk} is also not in \mathcal{H} . A similar argument implies that $S_{a_1 a_2 k}$ is also not in \mathcal{H} .

Under the transformations of the local subalgebra generated by $L_{ab'}$ and S_{-aij} we find that

$$\begin{aligned} \delta(\nabla x^a) &= -2\nabla y^{b'} \tilde{\Lambda}_{b'}^a + 2\nabla x_{ij} \Lambda_{+aij} + 2\nabla x^{b'} \Lambda_{b'}^a, \\ \delta(\nabla y_a) &= -2\nabla y^{b'} \Lambda_{b'a} + \epsilon_{ijkl} \Lambda^{+aij} \nabla_{kl} + 2\nabla x^{b'} \tilde{\Lambda}_{b'a}, \\ \delta(\nabla x^{a'}) &= 2(\nabla x_b - \epsilon_{bc} \nabla y^c) \Lambda^{ba'}, \delta(\nabla y^{a'}) = -2\epsilon_b^d (\nabla x_d - \epsilon_{dc} \nabla y^c) \Lambda^{ba'}, \\ \delta(\nabla x_{ij}) &= -(\nabla x^b - \epsilon^{bc} \nabla y_c) \Lambda_{+bij} \end{aligned} \quad (6.2.8)$$

We can derive these transformations from equations (6.1.8) provided we restrict the Λ parameters so that they belong to the subalgebra, that is, adopt the constraint $\Lambda_a^{b'} = \epsilon_{ac} \tilde{\Lambda}^{cb'}$ where $\tilde{\Lambda}^{cb'} = \Lambda^{cb'}{}_M q^M$ and the corresponding self-duality condition for Λ_{+aij} .

One can verify that the equations

$$\nabla_\alpha x^a - \epsilon^{ab} \nabla_\alpha y_b = 0 = \nabla_\alpha x^{a'} = \nabla_\alpha y^{a'} = \nabla_\alpha x_{ij} = 0 \quad (6.2.9)$$

are left invariant under the transformations of equation (6.2.8).

It is very instructive to analyse the equations (6.2.9) at the linearised level for which we assume static gauge $\partial_\alpha x^a = \delta_\alpha^a$ and so $\partial_\alpha y_a = -\epsilon_{\alpha a}$. From the expressions for the Cartan forms of equation (6.1.7) we find, without making any restrictions on the fields ϕ , that

$$\begin{aligned} \nabla_\alpha x^{a'} - \epsilon_\alpha^\beta \nabla_\beta y^{a'} &= \partial_\alpha x^{a'} - \epsilon_\alpha^\beta \partial_\beta y^{a'} + \dots, \nabla_\alpha x^{a'} + \epsilon_\alpha^\beta \nabla_\beta y^{a'} = \partial_\alpha x^{a'} + \epsilon_\alpha^\beta \partial_\beta y^{a'} + 4\hat{\phi}_\alpha^{a'} + \dots, \\ \nabla_\alpha x_{ij} &= \partial_\alpha x_{ij} - 4\phi_{-\alpha ij} + \dots \end{aligned} \quad (6.2.10)$$

The choice of local subalgebra \mathcal{H} can be deduced from the requirement that the above expressions for ϕ 's of equation (6.2.10) do occur in the group element g_h and that their

orthogonal combinations do not. Put another way it means that the orthogonal combinations occur in a local subalgebra transformation h and so can be removed from g_h . Since all the objects that occur in equation (6.2.10) vanish we find that half of the equations are inverse Higgs conditions and half are equations of motion.

The equations of motion can, following the arguments in section six, be written as

$$\sqrt{-\gamma}\gamma^{\alpha\beta}\nabla_\beta x^{\underline{a}} = -\epsilon^{\alpha\beta}\nabla_\beta y^{\underline{a}}, \quad \sqrt{-\gamma}\gamma^{\alpha\beta}\nabla_\beta x_{ij} = -\frac{1}{2}\epsilon^{\alpha\beta}\epsilon_{ijkl}\nabla_\beta x^{kl} \quad (6.2.11)$$

It would be interesting to analyse the content of these equations in detail. It would also be interesting to study the effect of the higher level local symmetries and the higher level conditions that they will impose.

We can count the number of bosonic degrees of freedom in this seven dimensional theory. We have $7 - 2 = 5$ degrees of freedom in $x^{a'}$ and $\frac{4 \cdot 3}{2 \cdot 2} = 3$ from x_{ij} which gives us 8 bosonic degrees of freedom. This is the number required for a half BPS brane that is maximally supersymmetric. In making this count we have assumed that the coordinates x^i, x_a^i, \dots do not contribute to the brane degrees of freedom.

6.3 The two brane

The two brane has a three dimensional world volume and so we take $a, b, \dots = 0, 1, 2$ and $a', b', \dots = 3, \dots, 6$. Following the same decompositions as in the previous sections, the rank two charge can be written as $Z^{a_1 a_2 M} = q^M Z^{a_1 a_2} + q_i^M Z^{a_1 a_2 i}$. We can choose the $Z^{a_1 a_2}$ charge to be the one that is active so breaking $SO(5)$ to $SO(4)$. We adopt similar decompositions of the generators and other objects as in the last section.

We choose our local subalgebra \mathcal{H} to contain the generators

$$\mathcal{H} = \{J_{ab}, J_{a'b'}, S^i_j, L_{ab'}, L_{ai}, S, \dots\} \quad (6.3.1)$$

where

$$L_{ab'} = J_{ab'} - \frac{1}{4}\epsilon_a^{e_1 e_2} S_{e_1 e_2 b'}, \quad L_{ai} = S_{ai} + \frac{1}{2}\epsilon_a^{e_1 e_2} S_{e_1 e_2 i}, \quad S = \frac{1}{3!}\epsilon_{a_1 a_2 a_3} S^{a_1 a_2 a_3} \quad (6.3.2)$$

These generators obey the algebra

$$[L_{ab'}, L_{cd'}] = 0, \quad [S_{ai}, S_{bj}] = 0, \quad [S, L_{ab'}] = 2L_{ab'}, \quad [S, L_{ai}] = 2L_{ai}, \dots \quad (6.2.3)$$

as well as the usual commutators with the Lorentz generators J_{ab} and $J_{a'b'}$.

Using the transformations of equation (6.1.8) but with the restrictions

$$\Lambda_{ab'} = 3\epsilon_a^{e_1 e_2} \Lambda_{e_1 e_2 b'}, \quad \Lambda_{ai} = \frac{1}{2}\epsilon_a^{e_1 e_2} \Lambda_{e_1 e_2 i}, \quad \Lambda_{a_1 a_2 a_3} = \frac{1}{3!}\epsilon_{a_1 a_2 a_3} \Lambda \quad (6.3.4)$$

so as to ensure that the transformation do belong to the local subalgebra, \mathcal{H} we find that

$$\delta(\nabla x^a) = 2\nabla x^{b'} \Lambda_{b'}^a - \epsilon_{ab_1 b_2} \Lambda \nabla x^{b_1 b_2} + 4\nabla x_i \Lambda^{ai} - 2\epsilon_{abc} \Lambda_{ci} \nabla x^{bi} + 4\epsilon_{abe} \Lambda_{ec'} \nabla x^{bc'}$$

$$\begin{aligned}
\delta(\nabla x^{a'}) &= 2\mathcal{E}^b \Lambda_b^{a'}, \quad \delta(\nabla x_i) = -\Lambda_{bi} \mathcal{E}^b, \quad \delta(\nabla x_{ai}) = -2\epsilon_{abc} \mathcal{E}^b \Lambda^c_i + \dots \\
\delta(\nabla x_{a_1 a_2}) &= 4\nabla x^{b'}_{[a_1} \Lambda_{a_2]b'} - 2\Lambda_{[a_1|i} \nabla x_{|a_2]}^i - 2\Lambda_{a_1 a_2 i} \nabla x^i + \epsilon_{a_1 a_2 b} \nabla x^b \\
\delta(\nabla x_{ab'}) &= -2\nabla x_{b'}^{c'} \Lambda_{c'a} - \Lambda_{ai} \nabla x_{b'}^i + \epsilon_{acd} \Lambda^d_{b'} \mathcal{E}^c
\end{aligned} \tag{6.3.5}$$

where $\mathcal{E}^a = \nabla x^a + \epsilon^a_{e_1 e_2} \nabla x^{e_1 e_2}$. Using these variations we find that

$$\delta(\mathcal{E}^a) = -2\Lambda \mathcal{E}^a - 2\Lambda_{ac'} \nabla x^{c'} \tag{6.3.6}$$

The local symmetries are preserved if we adopt the conditions

$$\mathcal{E}^a = \nabla_\alpha x^{a'} = 0 = \nabla_\alpha x_i \tag{6.3.7}$$

We now focus on the **linearised** theory and adopt the additional conditions

$$\nabla_{[\alpha} x_{a]c'} = 0, \quad \nabla_{[\alpha} x_{a]i} = 0 \tag{6.3.8}$$

provided we also demand certain higher level constraints. We note that we could have taken no antisymmetry on the above conditions and this would still preserve the symmetries. We have also taken only the lowest level brane world volume coordinate to be active.

To examine the meaning of these constraints we evaluate the Cartan forms at the linearised level. Using the lowest order conditions $\partial_\alpha x^a = \delta_\alpha^a$ and $\partial_\alpha x_{b_1 b_2} = \frac{1}{2} \epsilon_{\alpha b_1 b_2}$ and equation (7.1.7) we find taking the most general ϕ 's that

$$\nabla_\alpha x^{a'} + \epsilon_\alpha^{c_1 c_2} \nabla_{c_1} x_{c_2}^{a'} = 0, \quad \nabla_\alpha x^{a'} - \epsilon_\alpha^{c_1 c_2} \nabla_{c_1} x_{c_2}^{a'} = 8\Phi_{\alpha a'} \tag{6.3.9}$$

$$\nabla_\alpha x_i + \frac{1}{4} \epsilon_\alpha^{c_1 c_2} \nabla_{c_1} x_{c_2 i} = 0, \quad \nabla_\alpha x_i - \frac{1}{4} \epsilon_\alpha^{c_1 c_2} \nabla_{c_1} x_{c_2 i} = -4\Phi_{\alpha i} \tag{6.3.10}$$

where

$$\Phi_{\alpha a'} = \frac{1}{2}(\phi_{\alpha a'} - 3\epsilon_\alpha^{c_1 c_2} \phi_{c_1 c_2 a'}), \quad \Phi_{\alpha i} = \frac{1}{2}(\phi_{\alpha i} - \frac{1}{2} \epsilon_\alpha^{c_1 c_2} \phi_{c_1 c_2 i}) \tag{6.3.11}$$

We see that the conditions $\nabla_\alpha x^{a'} = 0$ and $\nabla_{[\alpha} x_{a]c'} = 0$ are half equations of motion and half inverse Higgs conditions. The same holds for the conditions $\nabla_{[\alpha} x_{a]i} = 0$ and $\nabla_\alpha x_i = 0$. The number of bosonic degrees of freedom are 4 for $x^{a'}$ and 4 for x_i giving a total of 8 bosonic degrees of freedom. This is the correct number for a maximally supersymmetric brane in a type II theory.

We now consider the **non-linear theory** and make a proposal along similar lines to that we have given for the M5 and D3 branes. We extend the world volume of the brane so that it has the coordinates $\xi^\alpha = \{\xi^\alpha, \xi_\alpha^i, \xi_{\alpha\beta'}\}$. and introduce the object $E_{\underline{\alpha}}^A = \nabla_{\underline{\alpha}} x^A$. where $x^A = \{x^a x_a^j, x_{ac'}\}$. We adopt the constraints of equation (6.3.7) but we replace the constraints of equation (6.3.8) by

$$\hat{\nabla}_{[a_1} x_{a_2]c'} = 0, \quad \hat{\nabla}_{[a_1} x_{a_2]i} = 0 \tag{6.3.12}$$

where

$$\hat{\nabla}_a = ((s^{-1})_a{}^\alpha + (\nabla_j^b x_a^j - \nabla^{bd'} x_{ad'}) (s^{-1})_b{}^\alpha) \nabla_\alpha \quad (6.3.13)$$

where $\nabla_A = (E^{-1})_A{}^\alpha \nabla_\alpha$. As before we vary so as to keep only terms with derivatives with respect to ξ^α and we used equation (6.35) in the form

$$\delta(\nabla x^a) = \Lambda \nabla x^a - 2\epsilon_{abc} \Lambda_{ci} E^{bi} + 4\epsilon_{abe} \Lambda_{ec'} E^{bc'} \quad (6.3.14)$$

and we have used the conditions of equation (6.3.7).

7. Some generic features of the brane dynamics

In this section we will discuss some of the generic features that emerge from formulating brane dynamics as a non-linear realisation of $E_{11} \otimes_s l_1$, as explained in section two. The dynamics of any brane is given in terms of the coordinates x^A , which are in one to one correspondence with the elements of the vector representation, and the fields φ arising from $I_c(E_{11})$ as well as background fields which arise from the Borel subalgebra of E_{11} . The charge of the brane being considered is one of the charges in the vector representation and the dynamics of the brane will involve the coordinate, corresponding to this charge, together with the coordinate x^a associated with usual spacetime translations P_a , as well as generically other coordinates.

It turns out that for every element in the Borel subalgebra of E_{11} there is an element in the l_1 representation [31]. At low levels this relation is one to one but at higher levels there is more than one element in the l_1 representation for each element in the Borel subalgebra of E_{11} . In the non-linear realisation every element in the Borel subalgebra of E_{11} leads to a background field and the brane couples to the field associated to the charge in the vector representation that it carries. To give a simple example, if we are considering the M2 brane its charge is $Z^{a_1 a_2}$, which is the second element in the vector representation in eleven dimensions, and the coordinates $x_{a_1 a_2}$ and x^a will play an important role in the dynamics of the M2 brane. The charge $Z^{a_1 a_2}$ corresponds in the Borel subalgebra of E_{11} to the generators $R^{a_1 a_2 a_3}$ and so the background field $A_{a_1 a_2 a_3}$ to which the M2 brane couples. These were indeed the features we found when we considered the dynamics of the M2 brane in section 4.1.

We will now illustrate, in outline only, some of the features of the dynamics of a p brane whose charge has one block of totally antisymmetrised spacetime indices. In this case the corresponding charge is of the form $Z^{a_1 \dots a_p \bullet}$ in the l_1 representation which leads to the corresponding coordinate $x_{a_1 \dots a_p \bullet}$ in the non-linear realisation where \bullet represents the internal group indices which we will suppress in what follows. We also have the coordinate x^a associated with the spacetime translations P_a . These occur in the Cartan form as follows;

$$\mathcal{V}_l = \nabla_\alpha x^a P_a + \dots + \nabla_\alpha x_{a_1 \dots a_p} Z^{a_1 \dots a_p} + \dots \quad (7.1)$$

Associated with the p-form charge is an element of the Borel subalgebra of E_{11} of the form $R^{a_1 \dots a_{p+1}}$ which obeys a commutator of the form

$$[R^{a_1 \dots a_{p+1}}, P_b] = f_1 \delta_b^{[a_1} Z^{a_2 \dots a_{p+1}] \quad (7.2)$$

where f_1 is a constant whose value is known from the E_{11} algebra. Associated with the generator $R^{a_1 \dots a_{p+1}}$ is a corresponding element of $I_c(E_{11})$ of the form $S^{a_1 \dots a_{p+1}}$ which has the following generic commutators

$$[S^{a_1 \dots a_{p+1}}, P_b] = f_1 \delta_b^{[a_1} Z^{a_2 \dots a_{p+1}]}, \quad [S^{a_1 \dots a_{p+1}}, Z^{b_1 \dots b_p}] = f_2 \delta_{b_1 \dots b_p}^{[a_1 \dots a_p} P^{a_{p+1}]} \quad (7.3)$$

where f_2 is a constant that is also specified by the E_{11} algebra. If we consider the indices on the generator $S^{a_1 \dots a_{p+1}}$ to take the values $a, b, \dots = 0, 1, \dots, p$, which are those in the brane direction, and define $S^{a_1 \dots a_{p+1}} \equiv -(p+1)! \epsilon^{a_1 \dots a_{p+1}} S$ then the above commutators become

$$[S, P_b] = f_1 \epsilon_{ba_1 \dots a_p} Z^{a_1 \dots a_p}, \quad [S, Z^{b_1 \dots b_p}] = f_2 \epsilon^{b_1 \dots b_p c} P_c \quad (7.4)$$

It will turn out that the generator S belongs to the local subalgebra \mathcal{H} and so it leads to a local symmetry of the non-linear realisation whose action is given in equation (2.15). As a result the Cartan forms of equation (6.1) transform as

$$\delta \mathcal{V}_l = -[\Lambda S, \mathcal{V}_l] \quad (7.5)$$

and using the commutators of equation (7.3) we find that

$$\delta(\nabla_\alpha x^a) = f_2 \Lambda \epsilon_{b_1 \dots b_p a} \nabla_\alpha x^{b_1 \dots b_p}, \quad \delta(\nabla_\alpha x_{a_1 \dots a_p}) = \Lambda f_1 \nabla_\alpha x^b \epsilon_{ba_1 \dots a_p} \quad (7.6)$$

An equation that is invariant under $SO(p+1) \otimes SO(11-p-1)$ Lorentz transformations, rigid E_{11} transformations, world volume diffeomorphism and the local transformations of equation (7.6) is given by is

$$\nabla_\alpha x^a = -e_1 \epsilon^{ab_1 \dots b_p} \nabla_\alpha x_{b_1 \dots b_p} \quad \text{equivalent to} \quad \nabla_\alpha x^{a_1 \dots a_p} = e_2 \epsilon^{ba_1 \dots a_p} \nabla_\alpha x_b \quad (7.7)$$

provided the constant $e_2 = \frac{1}{e_1 p!}$ and e_1 obeys the condition $e_1^2 = -(-1)^p f_2 f_1^{-1} (p!)^{-1}$. By varying this equation under the other transformations of the local subalgebra one can find at least some of the other equations of motion of the brane dynamics. Clearly one has to verify that the full set of equations of is invariant under all the symmetries of the non-linear realisation.

We can rewrite the second equation in (7.7) in the form

$$(s^{-1})_b{}^\alpha \nabla_\alpha x^{a_1 \dots a_p} = e_2 \epsilon^{ba_1 \dots a_p} \quad (7.8)$$

where $s_\alpha^b \equiv \nabla_\alpha x^b$. Using the identity

$$\epsilon^{\alpha_1 \beta_1 \dots \beta_p} s_{\alpha_1}^b s_{\beta_1}^{a_1} \dots s_{\beta_p}^{a_p} = (\det s) \epsilon^{ba_1 \dots a_p} \quad (7.9)$$

in equation (7.8) and multiplying by a further factor of s^{-1} we find the equation

$$\sqrt{-\gamma} \gamma^{\alpha\beta} \nabla_\beta x^{a_1 \dots a_p} = e_2 \epsilon^{\alpha\gamma_1 \dots \gamma_{p-1}} \nabla_{\gamma_1} x_{b_1} \dots \nabla_{\gamma_{p-1}} x_{b_{p-1}} \quad (7.10)$$

where

$$\gamma_{\alpha\beta} \equiv (s\eta s^T) = \nabla_\alpha x^a \eta_{ab} \nabla_\beta x^b \quad \text{and} \quad \gamma = \det \gamma_{\alpha\beta} = -(\det s)^2 \quad (7.11)$$

Using an identity similar to that of equation (7.9) we find that

$$\sqrt{-\gamma} \gamma^{\alpha\beta} \nabla_\beta x^a = e_1 \epsilon^{\alpha\beta\gamma_1 \dots \gamma_{p-1}} \nabla_\beta x^{ab_1 \dots b_{p-1}} \nabla_{\gamma_1} X_{b_1} \dots \nabla_{\gamma_{p-1}} X_{b_{p-1}} \quad (7.12)$$

The steps leading to equations (7.10) and (7.12) from equation (7.7) can be reversed and so both of these latter equations are equivalent to equation (7.7) and so also to each other. These equations are not the usual brane equations for a p-brane, but as we have explained for the branes we have studied in this paper we found that $\nabla_\alpha x^{a'} = 0$ and this allows us to extend the range on the indices in equation (7.12) from a, \dots to \underline{a}, \dots . One can then show that the φ fields then disappear from the equations and one finds the familiar brane dynamics at least for the coordinates $x^{\underline{a}}$.

In addition to the first order duality equations for the coordinates $x^{\underline{a}}$ and $x_{\underline{a}_1 \dots \underline{a}_p}$ there will be other dynamical equations involving the other coordinates of the non-linear realisation, that is, the vector representation. Since the full set of equations must be invariant under the symmetries of the non-linear realisation, which preserve the number of derivatives, the other equations must also be first order in derivatives and will also be duality equations that describe the world volume fields living on the world volume of the brane. This pattern is borne out for the branes we have studied in this paper.

We will now discuss how the above p-brane couples to the background fields. As we noted above the brane has a charge $Z^{\underline{a}_1 \dots \underline{a}_p}$ in the vector representation, associated with the coordinates $x_{\underline{a}_1 \dots \underline{a}_p}$, and related to the generators $R^{\underline{a}_1 \dots \underline{a}_{p+1}}$ in the Borel subalgebra of E_{11} which are in turn associated with the E_{11} background fields $A_{\underline{a}_1 \dots \underline{a}_{p+1}}$. Looking at our definition of the vielbein given below equation (2.9) and that the group element g_E contains the factor $e^{A_{\underline{a}_1 \dots \underline{a}_{p+1}} R^{\underline{a}_1 \dots \underline{a}_{p+1}}}$, we conclude that the vielbein has the component

$$E_{\underline{\mu}}^{\underline{a}_1 \dots \underline{a}_p} = -f_1 A_{\underline{\mu} \underline{a}_1 \dots \underline{a}_p} + \dots \quad (7.13)$$

and as a result

$$\nabla_\alpha^B x_{\underline{a}_1 \dots \underline{a}_p} = \partial_\alpha x_{\underline{a}_1 \dots \underline{a}_p} - f_1 \partial_\alpha x^\mu A_{\underline{\mu} \underline{a}_1 \dots \underline{a}_p} + \dots \quad (7.14)$$

We observe that this Cartan form, when its indices are in the brane world volume directions, and are totally antisymmetrised, is invariant under the gauge transformations

$$\delta x_{\underline{a}_1 \dots \underline{a}_p} = f_1 \Lambda_{\underline{a}_1 \underline{a}_2 \dots \underline{a}_p}, \quad \delta A_{\underline{a}_1 \dots \underline{a}_{p+1}} = \partial_{[\underline{a}_1} \Lambda_{\underline{a}_2 \dots \underline{a}_{p+1}]} \quad (7.15)$$

provided we adopt static gauge.

While the above parts of the brane dynamics and background coupling were worked out for a simple branes, they illustrate the general pattern that applies to all types of branes since the above results were derived from features of the $E_{11} \otimes l_1$ non-linear realisation that apply to all branes, that is, the relations between the brane charges in the l_1 representation and the generators in the Borel subalgebra which in turn implies the connections between the coordinates and the fields.

We have seen that we have a different choice of local subalgebra \mathcal{H} for each brane. This is to be expected as different branes break different parts of the E_{11} symmetry, which, of course, includes the Lorentz symmetry. We observe that the local subalgebra \mathcal{H} has the property that the generators of the vector representation decomposes under the action of the local subalgebra so as to contain a subrepresentation. In the cases we have studied the first element in this subrepresentation was

$$P_a + e_2 \epsilon_a^{b_1 \dots b_p} Z_{b_1 \dots b_p} \quad (7.16)$$

We recognise this as the constant part of the Cartan form \mathcal{V}_l in static gauge for which, using equation (7.7), is given by

$$\partial_\alpha x^a = \delta_\alpha^a, \quad \partial_\alpha x^{a_1 \dots a_p} = e_2 \epsilon^{\alpha a_1 \dots a_p} \quad (7.17)$$

The result means that the Cartan form \mathcal{V}_l has no constant term in its transformation under a local transformation.

It would be good to understand in a systematic way the correspondence between a chosen brane and its local subalgebra. One might imagine that that once one is given the brane charge in the vector representation one can deduce the corresponding local subalgebra by a well defined procedure. Given the choice of local subalgebra the dynamics of the brane are largely determined and so such an understanding may be important for understanding the general properties of branes.

8. Conclusion

In this paper we have further developed the theory of the non-linear of $E_{11} \otimes_s l_1$ to find brane dynamics. The brane moves through the spacetime which is automatically contained in the non-linear realisation and has coordinates in the vector representation. These contain the usual embedding coordinates as well as the world volume fields. The equations which emerge from the non-linear realisation are first order in derivatives and can be thought of as a set of duality equations. The dynamics for each brane is invariant under the full E_{11} symmetries, however, which parts are linearly realised and which non-linearly realised varies from brane to brane. This difference is reflected in the different choice of local subalgebra.

We have used this theory to find the dynamics of the strings in IIA and IIB theory, the M2 and M5 branes, the D3 brane in IIB as well as that for the one and two branes in seven dimensions. The construction of the linearised dynamics for all these branes is straightforward as well as for the non-linear theory for branes with world volume fields that do not carry Lorentz indices. However, our construction for the non-linear theory for the branes that contain world volume fields which carry Lorentz indices should be regarded as only a proposal. While some part of the dynamics of such branes is clear a problem arises with ensuring the gauge symmetries of such world volume fields. The proposed solution is to take the brane to depend on an enlarged world volume. This is natural in the sense that the brane moves not just in the usual Minkowski spacetime but also in the directions of the higher level coordinates that are automatically encoded in the E theory approach. These new brane coordinates ensure the gauge symmetry of these world volume

fields in a way that parallels the mechanism that is known to be present when the non-linear realisation is applied to find the field theories, that is, extensions of the maximal supergravities. However, our calculations in the brane case are limited and it would be good to extend them further and gain further evidence for this approach.

There should be no obstacle to using the non-linear realisation to construct the maximally supersymmetric brane dynamics at the linearised level in all dimensions. As we have mentioned, one will find a system of equations that are first order in derivatives acting on the coordinates of the vector representation. This implies that there is such a dual formulation for all branes. We note that the usual formulation of brane dynamics which follows from an action involves a derivative acting on the fields. As such their equations of motion will involve a derivative acting on an expression and one can find a formulation by integrating the equations of motion to find equations that are first order in derivatives and can be viewed as duality relation. Indeed one could derive these equations, independent of the E theory approach, where the dynamics is known.

In the brane dynamics we have constructed we only computed the dynamics for the lowest level coordinates and it remains to be found if there is further information in the higher level coordinates. If this were the case it could lead to features of brane dynamics not so far encountered. A similar remark applies to the use of the enlarged world volume.

In this paper we have not considered the construction of the Wess-Zumino terms and it would be interesting to consider this term from the viewpoint of the non-linear realisation. A discussion of this term using certain equations derived from E theory can be found in [29].

A different approach to brane dynamics based on current algebra was given in reference [32] and it would be interesting to understand the connection to the viewpoint of this paper.

As we have explained in the introduction, E_{11} , through its vector representation, predicts the existence of an infinite number of new branes almost all of which are exotic branes. The non-linear realisation discussed in this paper has the potential to give the dynamics of all these new objects.

The infinite number of new branes that E_{11} predicts in the vector representation could play an important role in understanding the entropy of black holes in the sense that they can be building blocks for a microscopic description of the entropy. Indeed one can wonder if the entropy is hidden in E_{11} and the vector representation.

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